Monetary Policy Arithmetic: Reconciling Theory with Evidence *

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Abstract

Empirical evidence indicates that in developed countries with low inflation rates, a permanent decrease in the inflation rate either has no impact on the capital stock and the level of output (superneutrality) or causes them to fall moderately. Existing budget arithmetic models of monetary policy cannot deliver superneutrality: they predict that a permanent decrease in the inflation rate will cause the capital stock and the level of output to rise, unless the economy is dynamically inefficient or is operating on the downward-sloping side of the inflation-tax Laffer curve. In this paper, we conduct a budget arithmetic analysis of monetary policy using a money demand specification – money in the utility function – that is new to this literature. We find that one simple, widely used assumption about utility from money delivers superneutrality, while a somewhat more general assumption delivers departures from superneutrality in the direction consistent with the evidence.

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1 Introduction

A permanent monetary policy shift has important repercussions for the government's budget. It affects the revenue from money creation (money seigniorage) and it is likely to change the cost of servicing the government debt. A complete description of the macroeconomic effects of permanent monetary policy change must take into account the impact of the fiscal policy adjustments the change may necessitate.

There is a fairly large literature that studies the real effects of permanent changes in monetary policy in overlapping generations models that explicitly incorporate the government budget constraint.[1] This "monetarist arithmetic" or "budget arithmetic" literature grew out of a seminal paper by Sargent and Wallace (1981) entitled "Some Unpleasant Monetarist Arithmetic." In budget arithmetic models, the government issues both money and bonds. Revenue from money creation covers the cost of debt service[2] plus a primary budget deficit. Since the primary deficit is assumed to be exogenous, changes in money seigniorage revenue caused by changes in monetary policy must be accompanied by changes in debt policy that have offsetting effects on the cost of debt service.[3]

The predictions of the budget arithmetic literature about the real effects of monetary policy are at variance with the empirical evidence in several important ways. The evidence suggests that in developed countries with low inflation rates, a permanent reduction in the inflation rate has either no long-run impact on the capital stock and output (superneutrality) or a moderately negative impact.[4] However, the budget arithmetic literature does not provide models that deliver superneutrality, and the departures from superneutrality are usually inconsistent with empirical

[1] A few papers that do not use OLG models are cited in Section 2.1.

[2] In this paper, the cost of debt service is the real interest the government pays on a fixed stock of debt. More generally, in a steady state with growth, it is the difference between the cost of paying the principal and interest on the debt from the previous period and the revenue from issuing new debt whose market value is equal to a constant fraction of real output. We focus on steady states where the gross real interest rate exceeds unity (our gross real growth rate), so that the cost of debt service is positive. In steady states where $R < 1$ (more generally, $R < n$, where n is the real growth rate), however, the revenue from issuing the new debt exceeds the cost of retiring the old debt, so that a policy of maintaining a fixed real stock of debt (more generally, a stock whose market value is a constant fraction of output) earns revenue for the government, on net. We refer to this revenue as "bond seigniorage."

[3] These assumptions seem quite realistic. Changes in taxes and spending require approval from the legislative branch. They are made relatively infrequently, and they are rarely coordinated with changes in monetary policy. In most cases, however, the executive branch can issue or retire debt almost autonomously, and the central bank is largely independent from the rest of the government. So the burden of adjustment to the budgetary impact of changes in monetary policy falls largely on the shoulders of the agency in the executive branch that conducts debt policy.

[4] Section 2.2 presents a detailed discussion of this evidence.

evidence: lower money growth and inflation produce a lower real interest rate[5] and higher levels of the capital stock and output. This result is often called "unpleasant monetarist arithmetic" (UMA). Models that deliver the opposite, empirically supported result – "pleasant monetarist arithmetic" (PMA) – can do so only when lower money growth and inflation increase revenue from money seigniorage, so that the initial equilibrium is on the "wrong" (downward-sloping) side of the inflation-tax Laffer curve,[6] or when the initial equilibrium is dynamically inefficient.[7]

The purpose of this paper is to construct a budget arithmetic model whose predictions are consistent with the empirical evidence on the long-run real effects of monetary policy. In particular, we provide a model that can deliver superneutrality from any initial real interest rate or inflation rate, under a widely-used money demand specification and a special but very common assumption about households' preferences. We also show that under a generalized version of the same preferences, a permanent reduction in the money supply growth rate may cause the real interest rate to rise and level of output to fall (the PMA result), even though the steady state is dynamically efficient and the economy is on the upward-sloping side of the inflation-tax Laffer curve. We provide a simple analytical characterization of the conditions under which this result holds.

More specifically, we use an overlapping generations (OLG) model with money in the utility function (MIUF) as our money demand specification. Although this money demand specification has been used frequently in models with infinitely-lived representative agents (ILRA models), it has never been used in overlapping generations branch of the budget arithmetic literature. We show that under this specification, money is superneutral as long as preferences for goods and money are separable and utility from money is logarithmic, which is the simplest and the most tractable parameterization of the utility from money. This is the first paper that obtains superneutrality in the budget arithmetic framework. Moreover, the budget arithmetic is in some sense the source of superneutrality. A reduction in the steady-state money supply growth rate has no impact on the capital stock and output, as long as the decrease in money seigniorage is exactly offset by the reduction in the government debt service cost for the constant real interest rate.

[5]The real interest rate in our model is defined as the net marginal product of capital. However, government bonds and productive capital are perfect substitutes in households' portfolios. Hence the rates of return on both assets are equal.

[6]See, for example, Bhattacharya *et al.* (1997).

[7]See, for example, Espinosa and Russell (1998) and Bhattacharya and Kudoh (2002). The empirical plausibility of dynamic inefficiency has been challenged by Abel *et al.* (1989). They compare capital income to investment in the United States and several other developed economies, and they conclude that all these economies are dynamically efficient.

It is well known that superneutrality holds in a large family of ILRA models with productive capital, money and government bonds (McCallum, 1990). Given that this is the case, why do we emphasize our superneutrality result? The reason is that in ILRA models the rate of time preference pins down the real interest rate in the steady state, making it policy-invariant. Thus, we do not get many insights about the sources of superneutrality, or departures therefrom, by studying ILRA models. In OLG models, on the other hand, permanent changes in monetary policy usually have the power to change the real interest rate – as the budget arithmetic literature (among others) demonstrates. So it is interesting to see why monetary policy may lose this power.

The literature also shows that overlapping generation models can deliver superneutrality if households care about the utility of their offspring and make positive bequests (Carmichael, 1982). This happens because intergenerational altruism and operative bequests effectively convert the OLG framework into an ILRA framework (Barro, 1974). However, there is a good deal of empirical evidence that the altruistic bequest motive is very weak, even for the OECD economies (Tachibanaki, 1994). Only accidental and precautionary bequest motives are empirically significant. We follow the rest of the OLG literature on budget arithmetic by assuming that households are selfish and do not make bequests.

Both our superneutrality result and our ability to produce pleasant monetarist arithmetic grow out of the fact that our money demand specification produces higher substitutability between money and other assets (productive capital and government bonds) than the specifications used in other budget arithmetic models. Our model follows the rest of the literature by requiring that the decrease in money seigniorage revenue caused by a policy-induced reduction in the steady-state inflation rate must be offset by a decrease in the cost of servicing the government debt. Under our money demand specification, however, a fall in the inflation rate produces a relatively large increase in the demand for real money balances. The increased money balances crowd out government bonds, which reduces the cost of debt service. If this effect is large enough, then there may be no need for a fall in the real interest rate to crowd out additional bonds by increasing the demand for capital, as in the rest of the literature. It may even be necessary for the real interest rate to rise, which will cause the capital stock and the level of output to fall (PMA).

The tendency for a decrease in inflation rate to cause the capital stock to fall, because agents substitute money for capital in their asset portfolios, is often referred to as the "Tobin effect" (Tobin, 1965). Weiss (1980) constructs an OLG model with money in the utility function that generates

a general equilibrium Tobin effect, but he does not model the government budget constraint in a convincing way. Our analysis can be characterized as integrating the Tobin-effect mechanism with the budget arithmetic mechanism, producing a range of outcomes that is broader and more realistic than the range either mechanism can produce alone. If the budget arithmetic effect dominates, then we get UMA; if the two effects exactly offset each other, then we get superneutrality; if the Tobin effect dominates, then we get PMA.

The rest of this paper is structured as follows. Section 2 reviews the literature on the budget arithmetic approach to monetary policy and summarizes the empirical evidence on long-run effects of changes in monetary policy. Section 3 describes the setup of the model and its competitive equilibria. The comparative statics results are presented in Section 4. Section 5 presents some concluding remarks.

2 Previous Literature

2.1 The Budget Arithmetic Approach to the Theory of Monetary Policy

The literature to which we contribute starts with the seminal work of Sargent and Wallace (1981).[8] They argue that the Fed's tight monetary policy (the Volcker disinflation) in the 1980s was not consistent with the Reagan's administration's loose fiscal policy, and might ultimately lead to higher rather than lower inflation. They argue that if the monetary authority reduces the money supply growth rate, then the fiscal authority is likely to lose revenue from currency seigniorage. If taxes and expenditures are not changed to offset this revenue loss, then the fiscal authority must cover it through increased borrowing. In addition, if the real interest rate on government debt is higher than the real growth rate of the economy,[9] then any increase in current borrowing necessitates increased future borrowing to cover the increased net interest cost. So the debt-to-GDP ratio begins to rise without bound. Eventually, the monetary authority will have to increase the money growth rate so that the entire net deficit (the primary budget deficit plus the net interest cost) can be covered by revenue from currency seigniorage. But since the government debt will be higher than it was before the policy change, the inflation rate will also have to be higher.

[8] The paper drew on ideas about the macroeconomic implications of the government budget constraint developed by Christ (1978, 1979).

[9] This was certainly true in the early 1980s.

Darby (1984) points out that "pleasant monetarist arithmetic" is also a strong possibility. He notes that for several decades prior to the 1980s the average real interest rate on government debt had been substantially lower than the average real output growth rate, making this seem like the normal relationship between the two variables. He argues that when the real interest rate is lower than the real growth rate, government debt produces revenue, on net;[10] an increase in the debt stock increases this revenue, *ceteris paribus*. Thus, the increase in the debt stock produced by the change in fiscal policy could provide the increase in net revenue needed to offset the decrease in revenue caused by the change in monetary policy. Miller and Sargent (1984) argue that the assumption that the real interest rate is fixed, made by Sargent and Wallace and followed by Darby, is not realistic. Instead, an increase in the debt stock could be expected to drive up the real interest rate on the debt. In that case, an increase in the debt stock could produce a loss of net revenue and an unsustainable debt spiral even if the initial real interest rate was lower than the output growth rate.

Wallace (1984) constructs a model in which the real interest rate is endogenous. He structures his model in a way that is consistent with unpleasant arithmetic by assuming that the real interest rate was higher than the output growth rate both before and after any change in policy. He avoids the dynamic complexity of Sargent and Wallace (1981) by focusing on steady states and by assuming that policy changes once, and only once, across steady states. Wallace (1984) departs from Sargent and Wallace (1981) by using the ratio of bonds to money (which was always constant in a steady state) as the monetary policy variable, instead of the money growth rate. An increase in the ratio of bonds to money is considered an open market sale and a monetary tightening, while an decrease is a purchase and an easing.[11]

When the "monetarist arithmetic" literature resumed in the late 1990s, [12] its participants were no longer exclusively interested in the effects of monetary policy on the inflation rate: they were

[10] See footnote 2

[11] This scheme for describing monetary policy allowed him to describe the experiment studied by Sargent and Wallace (1981) as the result of a single policy action: an open market sale, which increased the bonds-money ratio and produced a new steady state with a higher inflation rate.

Sargent and Wallace (1981) have also stimulated some studies that use ILRA models. Liviatan (1984) shows that Sargent and Wallace's(1981) result could be duplicated in an ILRA model. Drazen (1984) uses an ILRA model to show that if money demand was sufficiently elastic with respect to the inflation rate, then a monetary tightening might not lead to a debt spiral because it might cause currency seigniorage revenue to rise rather than fall. In both models, the real interest rate is invariant to changes in monetary policy – a characteristic of many monetary ILRA models.

[12] During 1985-1986, only two papers were added to the literature. These were Miller and Wallace (1985) and Miller and Todd (1995), who extended the analysis of Wallace (1984) to multi-country models.

also interested in its effects on the real interest rate and the level of output. For this reason, its participants have used OLG models rather than ILRA models. [13] Some of the later contributors have also followed Wallace (1984) by using the bonds-money ratio as their policy variable. In their models, an increase in this ratio – a monetary tightening – usually produces a higher real interest rate. If it also produces a higher inflation rate, then the model is said to display "unpleasant arithmetic." If it produces a lower inflation rate, then it displays "pleasant arithmetic."

This revived literature is organized into two strands. One strand attempts to defend the plausibility of "unpleasant arithmetic" by extending it to a broader range of environments and assumptions. The work of Bhattacharya, Guzman, Huybens and Smith (1997), Bhattacharya, Guzman and Smith (1998) and Bhattacharya and Kudoh (2002) are included in this strand.[14] The other strand views "pleasant arithmetic" as more plausible – sometimes referring to it as the "conventional wisdom" and looks for assumptions that will produce it. This strand includes Espinosa and Russell (1998, 2003).

Bhattacharya et al. (1997) use a model with neoclassical production and capital, following the lead of Miller and Sargent (1984). They focus on the case where the real interest rate exceeds the population growth rate. They find that, if the initial steady state is on the upward-sloping side of the inflation tax Laffer curve,[15] then an increase in the bonds-money ratio that produces a higher real interest rate also produces a higher inflation rate (unpleasant arithmetic) and a lower level of real output. If the initial steady state is on the downward-sloping side of the inflation tax Laffer curve, then a monetary tightening always produces pleasant arithmetic, i.e., the real interest rate rises and the inflation rate falls, as does the level of real output. Bhattacharya, Guzman and Smith (1998) adopt an endowment economy with a linear storage technology, so that the real storage return rate is fixed. This fixed return rate is assumed to exceed the population growth rate. They show that an increase in the bonds-money ratio, which they regard as a monetary tightening, always causes the inflation rate to rise, even when the real return rate on bonds is lower than the output growth rate. A peculiarity of their analysis is that the inflation rate always changes in opposite direction from the real interest rate on government bonds, which differs from the fixed storage

[13] An exception is Buffie (2003).

[14] Bhattacharya, Guzman, Huybens and Smith (1997) is a relatively ambitious paper with multiple messages. We include this paper in this survey because it makes the same assumptions about government budget constraint, and because it includes an investigation of the effects of changes in the bonds-money ratio whose results are characterized in terms of "unpleasant monetarist arithmetic."

[15] That is, if an increase in the inflation rate increases the revenue from currency seigniorage.

return rate because of a reserve requirement that is imposed on stored goods.[16] Bhattacharya and Kudoh (2002) return to neoclassical production assumptions of Bhattacharya et al. (1997). They focus on equilibria on the upward-sloping side of the inflation tax Laffer curve. They find that unpleasant arithmetic always holds if the real interest rate is higher than the population growth rate, and that it may hold if it is lower.

Turning to the other strand, Espinosa and Russell (1998) adopt the pure exchange model of Wallace (1984) to show that if the initial real interest rate is lower than the population growth rate and the bonds-money ratio is sufficiently low, then an increase in the ratio, which always causes the real interest rate to rise, will also cause the inflation rate to fall. Espinosa and Russell (2003) extend this analysis to a model of neoclassical production and capital as well as exogenous growth. They use the money growth rate (or, equivalently, the inflation rate) as their policy instrument, instead of the bonds-money ratio. They show that if the real interest rate is sufficiently low, so that the initial equilibrium is on the low-real-rate side of the Laffer curve for bond seigniorage, then an increase in the inflation rate always produces a decrease in the real interest rate and an increase in the level of real output. They also show that if intermediation costs drive a wedge between the marginal product of capital and the real interest rate on bonds, then it is possible to obtain results of this type even when the real return rate on capital exceeds the output growth rate. However, the initial steady states in question are still dynamically inefficient, because the real return rate on capital net of the intermediation cost is lower than the exogenous output growth rate.

As we have indicated, our analysis solves both of these problems. We use a money demand specification – money in the utility function – that is widely used in applied monetary theory but has not previously been used in the overlapping generations part of the monetarist arithmetic literature. We show that if we choose a particular but very widely used functional form for the utility from money (logarithmic), then our model delivers superneutrality for any initial policy setting – under our assumptions, any initial inflation rate – and any initial real interest rate. Second we show that for a more general functional form, with one parameter, there is a very wide range of parameter values for which we get pleasant monetarist arithmetic when the initial real interest rate is higher than the population growth rate and the initial steady state is dynamically efficient.

[16] A skeptic might argue that a monetary policy action that reduces the only endogenous real interest rate should be described as an easing, not a tightening. Under that interpretation, this model always delivers "pleasant arithmetic."

2.2 Long-Run Effects of a Change in the Inflation Rate: Empirical Evidence

Bullard and Keating (1995), King and Watson (1997), and Ahmed and Rogers (2002) analyze the impact of a permanent shock to the money growth rate, or the inflation rate, on the level of real output (GDP). Bullard and Keating (1995) use data from 58 developed and developing countries. They find that 16 of these countries have experienced both a permanent shock to the inflation rate and a permanent shock to the level of real output. Using the long-run identifying restriction that a real output level shock has no effect on the long-run inflation rate, they run a bivariate VAR for these countries. They find that the long run response of the level of output to a permanent inflation shock was positive and statistically significant for four countries and negative and statistically significant for one country; for the others, it was not significantly different from zero. They also find that in 9 countries there was no permanent shock to output, but a permanent shock to inflation occurred. This finding is taken as *prima facie* evidence of superneutrality. One additional finding of Bullard and Keating (1995) is that low-inflation countries respond to inflation shocks differently from high-inflation countries. In particular, for low-inflation countries the point estimate of the long-run response of output to inflation is generally positive, while for high-inflation countries it is zero or negative.[17]

King and Watson (1997) estimate a bivariate VAR of the money growth rate and the level of real output using U.S. postwar data. They test the hypothesis of long-run superneutrality across a broad range of possible identifying restrictions. They find that under most reasonable identifying restrictions, the hypothesis of superneutrality cannot be rejected. Although they also find various restrictions that are inconsistent with superneutrality, these rejections tend to be marginal and/or not robust to changes in the lag length or the sample-period. Ahmed and Rogers (2002) estimate a vector error correction model, using cointegrating relationships among the real output, inflation, consumption, investment, and the government share of GDP for U.S. data from 1889 to 1995. For two different specifications, their estimates indicate that a permanent shock to the inflation rate increases the levels of output, consumption, and investment.

Crosby and Otto (2000) analyze the impact on the capital stock of a permanent shock to the inflation rate, using the long-run identifying restriction that shocks to the capital stock do not have

[17] Our results provide a possible explanation for this finding. High-inflation countries may be on the unfavorable (downward-sloping) sides of their inflation-tax Laffer curves, while low-inflation countries may be on the favorable (upward-sloping) sides. If an economy displays PMA on the favorable side of its Laffer curve then it should display UMA on the other side.

permanent effects on the inflation rate. They construct capital stock series for 64 countries using post-WWII data. They find that 34 of these countries have had permanent shocks to both their inflation rates and their capital stocks. For these countries, they test the superneutrality hypothesis using a bivariate VAR. They find that for a large majority of the countries, a permanent inflation shock has no statistically significant impact on the capital stock in the long run.[18] When the superneutrality hypothesis is rejected, they find, in most cases, that the impact of higher inflation on the capital stock is positive.

Several studies analyze the long-run impact of inflation shocks on the real interest rate. In theoretical models with a neoclassical production function (including our model), there is a monotonic negative relationship between the real interest rate and the capital stock or the output level. Therefore, a negative effect of an inflation shock on the real interest rate can be interpreted as evidence of a positive impact of an inflation shock on the capital stock and output. On the other hand, evidence supporting the Fisher effect – invariance of the real interest rate to inflation rate – can be interpreted as evidence of superneutrality.

King and Watson (1997) test the long-run Fisher effect, with postwar U.S. data, using a bivariate VAR with CPI inflation and the nominal interest rate. They perform the test under a wide variety of identification schemes. They find many reasonable identifying restrictions that are consistent with the Fisher hypothesis. However, when statistically significant deviations from the Fisher relationship do occur, the nominal interest rate usually rises by less than the inflation rate, so that the impact of the inflation shock on the real interest rate is negative. Koustas and Serletis (1999) run similar bivariate VARs for 11 OECD countries. They reject the Fisher effect for most countries. They find that a positive inflation shock has less than a unit effect on nominal interest rate, and thus reduces real interest rate, even in the long run.

Rapach (2003) runs a trivariate VAR of the inflation rate, the nominal interest rate, and the level of output using postwar data for 14 OECD countries. Following Bullard and Keating (1995) and Crosby and Otto (1999), he uses long-run identifying restrictions developed by Blanchard and Quah (1989). For each country, the point estimate indicates that real interest rate falls in response to a permanent inflation shock. Moreover, the effects are often statistically significant at conventional significance levels. Rapach (2003) also finds that for a few countries, a permanent

[18] The sign of the effect of the inflation shock on the capital stock was positive for (almost) all the developed countries and many of the developing countries. The sign was negative for some developing countries, typically ones with high inflation.

inflation shock produces a statistically significant increase in the level of real output.

Finally, Rapach and Wohar (2004) test for multiple structural breaks in the mean real interest rate and mean inflation rate, using data for a number of OECD countries. For most countries, they find that the dates of the breaks in the inflation series and the real interest rate series are very near each another. They also find that increases in the mean inflation rate are always associated with decreases in the mean real interest rate, and vice-versa. They interpret their findings as suggesting that changes in monetary regimes bring about changes in real interest rates in the direction consistent with the pleasant monetarist arithmetic.

In summary, the evidence on long-run superneutrality is mixed: there is considerable evidence supporting it, but there is also some evidence against it. However, if long-run superneutrality does not hold, then monetary policy seems to have effects consistent with PMA, not UMA, at least for low-inflation countries. So the model that fits the evidence best would be a model that can deliver long-run superneutrality, but can also be specified so as to deliver PMA from steady states on the favorable (upward-sloping) side of the inflation tax Laffer curve. We lay out such a model in the next section of this paper.

3 The Model

3.1 The Environment

At each discrete date $t \geq 1$, a positive number of identical two-period-lived households (the members of generation t) are born. In period 1, there are also "initial old" households (the members of generation 0) who live for one period. The number of households per generation is constant: there is no population growth. Households have a time-separable utility function with standard properties. A household's lifetime utility depends positively on the real money balances it carries from its first period to its second period. This "money-in-the-utility-function" (MIUF) assumption can be justified as a reduced form of more explicit models of "shopping time" or "banking time."[19]

[19] The lower a household's real money balances, the more time it must spend shopping or banking (making trips to a bank to replenish cash balances), and the less time it has for leisure. McCallum and Goodfriend (1987) discuss the "shopping-time" justification for MIUF money demand. Croushore (1993) presents mathematical conditions for the equivalence of models with consumption and leisure in the utility function and models with consumption and money in the utility function.

We assume the following functional form for the utility function:

$$U_t = \frac{(c_{1,t})^{1-\theta} - 1}{1-\theta} + \beta \frac{(c_{2,t+1})^{1-\theta} - 1}{1-\theta} + \delta \frac{\left(m_t^d\right)^{1-\theta_m} - 1}{1-\theta_m}, \tag{1}$$

where $c_{1,t}$ is the consumption of the young household born in period t, $c_{2,t+1}$ is the consumption of the same household when it is old in period $t+1$, m_t^d is the real money balances that it has accumulated by the end of its first period of life (real money demand), β, δ, θ and θ_m are positive parameters.[20] Here β is the discount factor, δ shows the relative weight of the utility from holding money, θ and θ_m are parameters that govern the rate at which the marginal utility from consumption and the marginal utility from holding money balances, respectively, decrease as the relevant levels increase.[21]

Each two-period-lived household is endowed with one unit of labor in its first period of life and nothing in the second period. Hence, in the first period households work and earn wages. The labor market is perfectly competitive. In the second period, households live off of their savings. Leisure is not in the utility function, and work does not create disutility, so all households supply their unit of labor inelastically.

Households can choose among three different assets: productive capital, money, and government bonds. (If consumption goods are stored this period then they become capital goods next period.) Capital depreciates fully each period after use in production. Each initial old household is endowed with k_1 units of capital and M_0 units of nominal money.

Production of the single good is described by the production function $y_t = f(k_t) = Ak_t^{\alpha}$, where k_t is the per capita (per young household) capital stock and y_t is per capita output.

The government finances a purchase g per young household (assumed to be exogenous and constant) by printing money, issuing risk-free debt (government-backed one-period nominal bonds), or both. There are no taxes or transfers, so the government's primary budget deficit is equal to its purchases.

The trades in our model economy are as follows. Each period $t \geq 1$, young households are hired by competitive firms. The young households divide their wages (paid in goods) between first-period consumption, storage (to obtain capital next period) and purchases of money. Then they use some of their money to buy government bonds.

[20] This is a standard way to incorporate money balances in the utility function in an OLG model. See McCallum (1983) and McCallum (1987).

[21] In several parts of Section 4 we will assume that $\theta_m = \theta$.

The old households rent their capital to competitive firms, earning the marginal product of capital. Their second-period consumption is derived from their capital income and from the sale of the money they carried over from the preceding period or obtained by redeeming government bonds.

In period 1, all trades are the same as described above with one exception: the government does not redeem any bonds.

The nominal money stock evolves according to the equation:

$$M_{t+1} = \rho M_t,$$

where M_t is the nominal money stock per capita (per young household) at the end of period t, and $\rho > 0$ is the gross growth rate of the nominal money supply between periods t and $t+1$.

We define the real money supply per young household m_t^s as $\frac{M_t}{P_t}$, where P_t is the price level in period t. Hence the law of motion for the real money supply is:

$$m_{t+1}^s = \rho \frac{m_t^s}{\pi_t},$$

where π_t is the gross inflation factor between periods t and $t+1$, so that $\pi_t = \frac{P_{t+1}}{P_t}$.

The government's purchases are financed by money creation and the net proceeds of bond sales:

$$g_t P_t = M_t - M_{t-1} + B_t - B_{t-1} I_{t-1},$$

where B_t is the nominal per-young-household stock of one-period bonds issued in period t,[22] $I_{t-1} = R_{t-1}\pi_{t-1}$ is the gross nominal interest rate on bonds issued in period $t-1$, and R_{t-1} is the real gross return on these bonds. By arbitrage it is equal to the return on capital, i.e.

$$R_{t-1} = r_t = f'(k_t) = A\alpha k_t^{\alpha-1}$$

The government budget constraint can be written in real terms as:

$$g_t = \frac{M_t - M_{t-1}}{P_t} + b_t - b_{t-1}R_{t-1} = (1 - \frac{1}{\rho})m_t + b_t - b_{t-1}R_{t-1}, \tag{2}$$

where $b_t = \frac{B_t}{P_t}$ is the real stock of bonds.

[22] B_t is the market value of the bonds issued at t, not the face (payoff) value.

3.2 Competitive Equilibria

The two-period-lived households maximize their utility (1) subject to the budget constraints[23]

$$c_{1,t} = w_t - m_t^d - k_{t+1} - b_t \tag{3}$$

$$c_{2,t+1} = \frac{m_t^d}{\pi_t} + R_t(k_{t+1} + b_t). \tag{4}$$

The gross rate of return on money is:

$$R_t^m = \frac{1}{\pi_t}.$$

In a steady state $R_t^m = \frac{1}{\rho}$.

Households maximize their utility with respect to real money balances, m_t^d, holdings of capital, k_{t+1}, and bond holdings, b_t. However, they are indifferent between bonds and capital when the expected return rates on these assets are equal. Therefore, they effectively optimize over two variables: total holdings of the capital market assets, $s_t \equiv k_{t+1} + b_t$, and m_t^d. Budget constraints (3) and (4) can be combined into an intertemporal budget constraint:

$$c_{1,t} + \frac{c_{2,t+1}}{R_t} = w_t - m_t^d \left(1 - \frac{R_t^m}{R_t}\right).$$

Optimization results in the first-order conditions:

$$-\frac{1}{(c_{1,t})^\theta} + \beta R_t \frac{1}{(c_{2,t+1})^\theta} = 0 \tag{5}$$

and

$$-\frac{1}{(c_{1,t})^\theta} + \beta R_t^m \frac{1}{(c_{2,t+1})^\theta} + \delta \frac{1}{(m_t^d)^{\theta_m}} = 0. \tag{6}$$

Combining (5) and (6) yields:

$$R_t = R_t^m + \frac{\delta\,(c_{2,t+1})^\theta}{\beta\,(m_t^d)^{\theta_m}}. \tag{7}$$

In equilibrium, there is a wedge between the return rates on money and capital. Since money enters the utility function directly and the functional form does not allow for satiation, households are

[23] The initial old maximize:

$$U_0 = \beta \frac{(c_{2,1})^{1-\theta} - 1}{1-\theta} + \gamma \frac{(m_0^d)^{1-\theta_m} - 1}{1-\theta_m}$$

subject to:

$$c_{2,1} = \frac{M_0}{P_1} + k_1 R_0 \qquad (4').$$

willing to hold positive money balances when the rate of return on money is lower than the return rate on capital and bonds. This condition, $R > R^m$ is necessary for finite money demand and positive demand for nonmonetary assets. The magnitude of the wedge depends, in part, on the preference parameters β and δ. The larger δ is relative to β, i.e. the more intense the household's preference for money balances vis-a-vis second-period consumption, the larger the wedge between R and R^m it is willing to tolerate.

For the special case $\theta = \theta_m$ it is easy to show, using the budget constraints (3) and (4) and the first-order conditions (5) and (6), that the elasticities of the money-to-consumption ratios $\frac{m_t^d}{c_{1,t}}$ and $\frac{m_t^d}{c_{2,t+1}}$ with respect to R^m are equal to $\frac{1}{\theta_m}\frac{R_t^m}{R_t - R_t^m}$. Thus as θ_m gets smaller, households are more willing to substitute real money balances for consumption as the inflation rate falls, other things equal. And since households hold capital and bonds only to secure second-period consumption, it follows that they will substitute real money balances for these assets.

Factor markets are perfectly competitive. Both labor and capital earn their marginal products. Hence,

$$r_t = A\alpha k_t^{\alpha-1} \tag{8}$$

and

$$w_t = A(1-\alpha)k_t^{\alpha}. \tag{9}$$

The money market equilibrium condition,

$$m_t^d = m_t^s, \tag{10}$$

completes the description of the competitive equilibrium.

We conclude the subsection by defining formally the competitive equilibrium.

Definition: A competitive equilibrium consists of a value $\rho > 0$, a sequence b_t for $t = 1, 2, ...$, non-negative sequences $s_t, , c_{1,t}, c_{2,t}$ for $t = 1, 2, ...$ and k_t for $t = 2, 3, ...$, positive sequences $m_t^d, m_t^s, P_t, r_t, \pi_t, w_t$ for $t = 1, 2, ...$ and R_t for $t = 0, 1, ...$ such that, given $g_t = g > 0$ for $t = 1, 2, ...$ and the initial conditions $M_0 > 0, k_1 > 0, B_0 = 0$

(a) factor markets clear, i.e., (8) and (9) hold for $t = 1, 2, ...$;

(b) asset markets clear, i.e., $s_t = b_t + k_{t+1}$ for $t = 1, 2, ...$;

(c) money market clear, i.e., $m_t^s = m_t^d$ for $t = 1, 2, ...$;

(d) $P_t > 0$ for $t = 1, 2, ...$; $\pi_t = \frac{P_{t+1}}{P_t}$ for $t = 1, 2, ...$;

(e) $R_t^m < R_t$, where $R_t^m = \frac{P_t}{P_{t+1}}$, for $t = 1, 2, ...$ and $R_{t-1} = r_t$ for $t = 1, 2, ...$;

(f) $m_{t+1} = \frac{\rho m_t^s}{\pi_t}$ for $t = 1, 2, ...$ and $m_1 = \frac{M_0}{P_1}\rho$;

(g) the two-period-lived households choose their consumption and assets optimally, i.e., (5) and (7) hold for $t = 1, 2, ...$;

(h) the budget constraints of the households are satisfied, i.e., (3) and(4) hold for $t = 1, 2, ...$ and $(4')$ holds for $t = 0$;

(i) the government's budget constraint (2) is satisfied in periods $t = 2, 3, ...$, and in period 1, $gP_1 = M_1 - M_0 + B_1$.

To simplify the notation, we will drop the real-money-balance superscripts from this point forward, using m_t for both real money demand and real money supply.

3.3 Steady States

A steady state satisfies the competitive equilibrium conditions for constant values of the endogenous real variables. Conditions that apply only at date 1 are ignored, as are variables that appear only at date 1. The nine endogenous variables in a steady state are the household choice variables c_1, c_2, s and m, the firm choice variable k, the government policy variable b, and the market wage and return rates w, R and R^m. We think of the government as choosing b to accommodate its choice of ρ, the active policy variable.

The competitive steady state equilibrium is determined by the following set of equations:

$$w = A(1-\alpha)k^{\alpha} \tag{11}$$

$$R = A\alpha k^{\alpha-1} \tag{12}$$

$$c_1 = w - k - b - m \tag{13}$$

$$c_2 = (k+b)R + mR^m \tag{14}$$

$$c_2 = (\beta R)^{1/\theta} c_1 \tag{15}$$

$$R = R^m + \frac{\delta c_2^{\theta}}{\beta m^{\theta_m}} \tag{16}$$

$$g = (1 - R^m)m + b(1 - R) \tag{17}$$

$$s = b + k \tag{18}$$

$$R^m = \frac{1}{\rho} \tag{19}$$

Equations (11) and (12) determine the factor prices: the return on capital and the wage rate. Equations (13) and (14) are the budget constraints of households when they are young and when they are old, respectively. Equations (15) and (16) are the first-order conditions of the household's intertemporal optimization problem. Equation (17) is the government budget constraint. Equation (18) is the asset market clearing condition. Finally, equation (19) says that in a steady state, the gross rate of return on money is the inverse of the gross money supply growth rate.

Analysis of the existence of steady states is relegated to the appendices. Appendix A derives existence conditions for the case $\theta = \theta_m = 1$. Appendix B presents numerical examples of steady states when θ and θ_m are greater or less than unity.[24] We construct these examples in the following way. We select a set of values for A, α, β, ρ, $\frac{g}{y}$[25] that are used in business cycle literature and/or are comparable to those observed in modern economies. We combine them with six pairs of values of θ, and θ_m, and search for the set of values of δ compatible with the real solution to the system (11)-(19).

There may be more than one steady state associated with a given value of ρ. Proposition 1 below applies to any steady state. Proposition 2 applies only to steady states that satisfy certain conditions.

4 Comparative Statics

The comparative statics analysis proceeds in two steps. First, we derive conditions for superneutrality of money, i.e., conditions under which a change in the steady-state money growth rate has no impact on the steady state levels of the capital stock, consumption, or output. Second, we analyze departures from superneutrality, i.e., we describe conditions under which we can show that a reduction in the steady-state money growth rate affects the capital stock, consumption, and output in particular ways (PMA or UMA). The proofs are presented in Appendix C.

[24] In most budget arithmetic models, a household makes only one non-trivial decision in its first period of life: how much to save and how much to consume. In our model it faces a dual decision: how to divide its wage income between consumption and saving, and how to divide its saving between capital/bonds and money. This extra dimension makes the formal analysis more complicated.

[25] Note that $y = Ak^{\alpha}$.

4.1 Superneutrality of Money

Proposition 1 *Suppose a particular value of ρ supports a steady state. Then a marginally different value of ρ supports a steady state with unchanged values of all real variables except R^m, m and b if and only if $\theta_m = 1$.*

Notice that Proposition 1 does not impose any restrictions on θ, the curvature parameter for utility from consumption. Furthermore, it holds whether R is greater or less than unity (whether the steady state is dynamically efficient or inefficient).

Under money demand assumption we use here, a decrease in ρ (an increase in R^m) always causes household money demand to increase. If the credit market is to clear at unchanged values of R, k and c_1, then this increase must be matched by an equal decrease in b, the real stock of government debt per young household. But this decrease in b is consistent with equilibrium only if the government budget constraint is satisfied. The increase in R^m causes the government to lose revenue from money seigniorage, despite the increase in m. The decrease in b reduces the interest cost of rolling over the government debt. When $\theta_m = 1$, these two revenue effects offset each other exactly, so that there is a new steady state with higher values of R^m and m but unchanged values of all other real variables except b, which falls by the same amount m rises.

4.2 Departures from Superneutrality

A complete comparative statics analysis of system (11)-(19) would be quite complicated. To analyze the impact of a change in the money supply growth rate on the capital stock and output, we proceed in three steps. First, we define the government revenue function

$$G(R, \rho) = (1 - \frac{1}{\rho})m(R, \rho) + (1 - R)b(R, \rho).$$

We are now treating G as an endogenous variable in system (11)-(19), and R as exogenous. Here $m(R, \rho)$ is the household money demand function and $b(R, \rho)$ is the equilibrium supply-of-bonds function that comes from the household money demand function, the young household budget constraint (13) and the capital demand function $k(R)$, which is obtained by inverting condition (12). In other words, we redefine the system in order to be able to study how government revenue G changes, across steady states, following exogenous changes in R and ρ. First, we study the relationship between ρ and G when R is held constant (Lemma 1). Next, we study the relationship

between R and G when ρ is held constant (Lemma 2). Finally, we use these results to determine how R, the return rate on the capital market assets, must change, following a change in ρ, in order to keep government revenue constant, so that $G(R, \rho) = g$ (Lemma 3 and Proposition 2).

Lemma 1

$$\frac{\partial G}{\partial \rho} \gtreqless 0 \quad \textit{if and only if} \quad \theta_m \gtreqless 1.$$

In words, Lemma 1 asserts that for given values of k and R, a permanent reduction in the rate of money supply growth lowers the government revenue if $\theta_m > 1$, increases the government revenue if $\theta_m < 1$, and leaves it unchanged if $\theta_m = 1$.

Lemma 2 *Suppose $\theta_m = \theta$. If a particular value of ρ supports a steady state with $R > 1$ and $b > 0$, then $\frac{\partial G}{\partial R} < 0$ across steady states with the same value of ρ.*

Lemma 2 asserts that in a dynamically efficient economy ($R > 1$), holding $R^m = 1/\rho$ constant, an increase in the rate of return on the capital market assets, R, decreases the government revenue. Even though the proof of the Lemma is quite complicated, the intuition is straightforward: this change in R raises the cost of servicing the government debt, $b(R-1)$, and hence reduces the net government revenue, $G = m(1-1/\rho) - b(R-1)$.[26] Furthermore, the last expression for G illustrates the importance of the condition $b > 0$. If this condition is not satisfied, the direct effect of R on G (i.e., without taking into account the endogenous changes in m and b) will be reversed.

Furthermore, conditions $R > 1$ and $b > 0$ are quite plausible empirically. The modern developed economies are dynamically efficient (Abel, et al., 1989), and in all of them (except Norway), the stock of government debt exceeds the government holdings of financial assets.

Lemma 3 *If $\frac{\partial G}{\partial R} < 0$ from an initial steady state, then $\frac{\partial k}{\partial \rho} \lesseqgtr 0$ as $\theta_m \gtreqless 1$.*

Lemma 3 puts it all together: if a reduction in ρ causes G to rise, and an increase in R causes G to fall, and if G is to remain fixed then both ρ and R must change in the opposite directions. This happens if $\theta_m < 1$. Hence, ρ and k must move in the same direction, because k is inversely related to R along the firms' capital demand function. If $\theta_m > 1$, then ρ and k must move in the opposite directions.

[26] The proof is quite complicated, because m and b are endogenous. However, the "direct" effect of a change in R dominates, and the government revenue is inversely related to R.

Proposition 2 below describes the effects of a change in the money supply growth rate in cases where the superneutrality of money does not hold. In order to make the analysis tractable, we study the special case $\theta_m = \theta$. We also need to assume that the stock of government debt is positive ($b > 0$) and that the steady state is dynamically efficient ($R > 1$).

When $\theta_m < 1$ we find that the departure from superneutrality is always in the "Keynesian" or PMA direction, which is the direction the empirical evidence seems to support. If $\theta_m > 1$, but sufficiently close to 1, then the departure is always in the UMA direction.

Proposition 2 *Suppose $\theta = \theta_m$. If a particular value of ρ supports a steady state with $R > 1$ and $b > 0$, then 1) $\frac{\partial k}{\partial \rho} > 0$ if $\theta < 1$; 2) there exists $\bar{\theta} > 1$ such that for any $\theta \in (1, \bar{\theta})$, $\frac{\partial k}{\partial \rho} < 0$.*

Proposition 2 establishes sufficient, but not necessary, conditions for the UMA and PMA deviations from superneutrality. The following numerical examples illustrate that equality of θ and θ_m is not necessary for these conditions to deliver UMA or PMA. In these examples, PMA holds whenever θ_m is less than unity and UMA holds whenever θ_m is greater than unity. We have not been able to construct counterexamples, so we suspect that Proposition 2 holds even when θ and θ_m are different and/or when the side condition for the second part of the proposition fails.

For all the examples, we set $A = 1, \alpha = \frac{1}{3}, \beta = 1, \delta = \frac{1}{3}$, and $g = 0.001$. The monetary policy experiment is a reduction of the gross nominal money supply growth rate from $\rho_1 = 1.3$ to $\rho_2 = 1.2$.

Table 1. Change in k following a permanent monetary contraction

	Initial value of k	Terminal value of k
$\theta = \theta_m = 1.5$	0.0516	0.0522
$\theta = 1.5$; $\theta_m = \frac{2}{3}$	0.1413	0.1396
$\theta = \frac{2}{3}$; $\theta_m = 1.5$	0.0359	0.0363

Lines 1 and 3 show the cases when $\theta_m = 1.5 > 1$. In these cases, a permanent monetary contraction raises the capital stock, which is UMA. Line 2 shows the case when $\theta_m = 2/3 < 1$. In this case, a permanent monetary contraction reduces the capital stock, which is PMA.

4.3 Discussion of the Results

Previous research using OLG models to pursue the budget arithmetic approach to monetary policy has used one of two money demand specifications: reserve requirements and random relocation. Under these specifications, money demand is either completely insensitive (reserve requirements) or very insensitive (random relocation) to changes in the inflation rate.[27] As we have seen, a reduction in ρ, and the resulting increase in the return on money R^m, reduces government revenue from money creation, $(1 - R^m)m$. This reduction must be offset by a decrease in the cost of debt service. If the real interest rate is to remain unchanged, then debt service costs can fall only if the stock of government bonds falls: this can happen if money crowds bonds out of households' asset portfolios. But since money demand is relatively insensitive to changes in R^m, it will not rise substantially, so it will not crowd out enough bonds to produce the required cost decrease. Additional bonds will need to be crowded out by capital, which will require a decrease in the real interest rate. This result – that monetary tightening causes the capital stock to rise and the real interest rate to fall – is inconsistent with the empirical evidence.

The model developed in this paper overcomes the problem just described by using a money-in-the-utility-function specification, which allows us to make money demand much more inflation-sensitive. If $\theta = \theta_m$, then, as we show in Section 3, the elasticities of money-to-consumption ratios, $\frac{m}{c_1}$ and $\frac{m}{c_2}$ with respect to R^m are equal to $\frac{1}{\theta_m}\frac{R^m}{R-R^m}$. This elasticity is always positive, and it is inversely related to the value of θ_m. Households are now able and willing to substitute money for capital market assets after a permanent disinflation – particularly when θ_m is relatively low – even though these assets remain superior to money as pure stores of value. Varying θ_m allows us to study how the inflation sensitivity of money demand, and thus the degree of substitutability between real money balances and capital market assets, affects the direction of the impact of monetary tightening on the capital stock, the real interest rate, and the level of output. If θ_m is greater than unity then money demand rises only slightly in response to a disinflation, relatively

[27] Espinosa and Russell (1998, 2003) induce money demand with a binding reserve requirement. Under this money demand specification, a change in the inflation rate cannot produce any direct substitution between money and capital market assets. In Bhattacharya et al. (1997), a fraction of the young households are relocated, at the end each period, to a location at which they can obtain goods to consume next period only by purchasing them with money. Since the households are risk averse, and failure to consume in the second period produces a low level of utility, they are reluctant to reduce their money holdings even when the inflation rate rises. If the households maximize their lifetime expected utility using the logarithmic utility function $U = \ln(c_{1,t}) + \beta \ln(c_{2,t+1})$, for example, then their demand for real money balances is completely invariant to the real return rate on money, just as in the case of a reserve requirement.

few bonds are displaced by money, and "unpleasant monetarist arithmetic" holds. If θ_m is smaller than unity, then money demand rises substantially, a large volume of bonds is replaced by money, and we have "pleasant monetarist arithmetic." Finally, if $\theta_m = 1$, money demand rises just enough to crowd out the quantity of bonds that will cause reduced debt service costs to offset the loss of revenue from money seigniorage, so that the real interest rate, the capital stock, and the level of output are not affected. In this case, money is superneutral.

5 Concluding Remarks

The paper extends the literature on monetary policy budget arithmetic in overlapping generations models in a way that that [1] reconciles its predictions with the empirical evidence about the real effects of monetary policy in developed, low-inflation economies and [2] is consistent with two widely held views about the characteristics of the competitive equilibria in these economies. Our first result describes conditions under which monetary policy is long-run superneutral, so that a permanent monetary tightening has no effect on the steady state value of the real interest rate, the capital stock and the level of output. This result is new to the OLG literature on budget arithmetic, and it is consistent with a wealth of empirical evidence that supports long-run superneutrality. Our second result describes conditions under which a permanent monetary tightening causes the steady state real interest rate to rise and the capital stock and the level of output to fall. This result is consistent with empirical evidence which suggests that if superneutrality does not hold, then departures from it are likely to be in this "pleasant arithmetic" direction, particularly for developed countries with relatively low inflation rates. Although other papers in the literature have delivered pleasant arithmetic, ours is the first to do so from initial steady states that are both dynamically efficient and on the favorable sides of their economies' inflation-tax Laffer curves. Most macroeconomists believe that competitive equilibria in developed, low-inflation countries have both these characteristics.

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Appendix

A. Existence of Equilibrium in the Case of Logarithmic Preferences

This section derives the existence conditions for a particular case of the logarithmic preferences: $\theta = \theta_m = 1$. In other words, the utility function is:

$$U = \ln(c_1) + \beta \ln(c_2) + \delta \ln(m)$$

By eliminating R^m, w and s from the system (11)-(19) and replacing the second-period budget constraint (14) with the national accounts identity $Ak^\alpha = c_1 + c_2 + g + k$[28], we rewrite the steady-state equilibrium conditions (11)-(19) in the following way:

$$\begin{aligned}
y &= c_1 + c_2 + g + k && (20)\\
y &= Ak^\alpha && (21)\\
R &= A\alpha k^{\alpha-1} && (22)\\
c_2 &= \beta R c_1 && (23)\\
R &= \frac{1}{\rho} + \frac{\delta c_2}{\beta m} && (24)\\
c_1 &= Ak^\alpha - kR - k - b - m && (25)\\
g &= (1 - \frac{1}{\rho})m + b(1 - R) && (26)
\end{aligned}$$

Proposition A1 states conditions for the existence of the steady- state equilibrium described by the system (20)-(26).

Proposition A1: **The steady-state equilibrium described by the system (20)-(26) exists if and only if:**

$$\begin{cases}
\boldsymbol{L(k_0) \geq R(k_0)} \\
\boldsymbol{\alpha^2(\alpha+\beta+\delta)^2 - 4(1+\beta+\delta)(1-\alpha)\alpha\beta(2\alpha-1) \geq 0} \\
\boldsymbol{k_0 < \tilde{k}, \quad \text{or} \quad L(\tilde{k}) > R(\tilde{k})}
\end{cases}$$

[28] Of the four constraints (the two private agent's budget constraints, the government budget constraint, and the goods market clearing condition) any one of them is redundant. It can be written as a linear combination of the others.

where $\tilde{k} = (A\alpha\rho)^{1-\alpha}$, $L(k) = \frac{(\alpha+\beta+\delta)Ak^{\alpha}}{1+\beta+\delta}$, $R(k) = \frac{(1-\alpha)A^2\alpha\beta k_0^{2\alpha-1}}{1+\beta+\delta} + g + k$, and

$$\left[\begin{array}{ll} k_0 = \left(\frac{-\alpha(\alpha+\beta+\delta)+\sqrt{\alpha^2(\alpha+\beta+\delta)^2+4(1+\beta+\delta)(1-\alpha)\alpha\beta(1-2\alpha)}}{2(1-\alpha)\alpha\beta(1-2\alpha)A}\right)^{1/(\alpha-1)} & \textbf{for} \quad \alpha \neq 0.5 \\ k_0 = \left[\frac{(\alpha+\beta+\delta)A}{2(1+\beta+\delta)}\right]^2 & \textbf{for} \quad \alpha = 0.5 \end{array}\right.$$

Proof: Successively eliminating c_1, c_2, R, m, b and y from the system (20)-(26), one obtains:

$$\frac{(\alpha+\beta+\delta)A}{1+\beta+\delta}k^{\alpha} = \frac{(1-\alpha)\alpha\beta A^2}{1+\beta+\delta}k^{2\alpha-1} + g + k \tag{27}$$

Denote the left-hand side of equation (27) as $L(k)$, and the right-hand side as $R(k)$. Their derivatives are: $L'(k) = \frac{(\alpha+\beta+\delta)A\alpha}{1+\beta+\delta}k^{\alpha-1}$ and $R'(k) = \frac{(1-\alpha)\alpha\beta A^2(2\alpha-1)}{1+\beta+\delta}k^{2\alpha-2} + 1$. Equilibrium exists if and only if the equation $R(k) = L(k)$ has a nonnegative solution. Three cases, when $\alpha < 0.5$, $\alpha > 0.5$, and $\alpha = 0.5$, are considered separately.

Case 1. $\alpha < 0.5$. In this case $2\alpha - 1 < 0$ and the function $R(k)$ approaches $+\infty$, when k approaches zero. Its derivative $R'(k)$ is monotonically increasing. It is negative for small values of k, and positive for large values of k. $R(k) > L(k)$ for small values of k, but the difference is falling. It attains the minimum at k_0, the root of the equation

$$L'(k) = R'(k) \tag{28}$$

and grows back afterwards. This case is depicted on figure 1.

Insert figure 1 here.

Equilibrium exists if and only if

$$L(k_0) \geq R(k_0) \tag{29}$$

Equation (28) is a quadratic equation in $k^{\alpha-1}$. The only positive solution is:

$$k_0^{\alpha-1} = \frac{-\alpha(\alpha+\beta+\delta) + \sqrt{\alpha^2(\alpha+\beta+\delta)^2 + 4(1+\beta+\delta)(1-\alpha)\alpha\beta(1-2\alpha)}}{2(1-\alpha)\alpha\beta(1-2\alpha)A} \tag{30}$$

Case 2. $\alpha > 0.5$. In this case $2\alpha - 1 > 0$. The function $R(k)$ is monotonically increasing, but its derivative is not. Equation (28) has two positive roots:

$$k_2^{\alpha-1} = \frac{\alpha(\alpha+\beta+\delta) - \sqrt{\alpha^2(\alpha+\beta+\delta)^2 - 4(1+\beta+\delta)(1-\alpha)\alpha\beta(2\alpha-1)}}{2(1-\alpha)\alpha\beta(2\alpha-1)A} \tag{31}$$

and

$$k_1^{\alpha-1} = \frac{\alpha(\alpha+\beta+\delta) + \sqrt{\alpha^2(\alpha+\beta+\delta)^2 - 4(1+\beta+\delta)(1-\alpha)\alpha\beta(2\alpha-1)}}{2(1-\alpha)\alpha\beta(2\alpha-1)A} \tag{32}$$

Needless to say, these roots are real and positive only if the discriminant of the quadratic equation is nonnengative, or

$$\alpha^2(\alpha+\beta+\delta)^2 - 4(1+\beta+\delta)(1-\alpha)\alpha\beta(2\alpha-1) \geq 0$$

It is straightforward to verify that the derivative of $R(k)$ is greater than the derivative of $L(k)$ for $k < k_1$ and $k > k_2$, and smaller for $k_1 < k < k_2$. $R(k) > L(k)$ for small values of k, and the difference rises until $k = k_1$. Then

$R(k) - L(k)$ falls until $k = k_2$, and rises again afterwards to become infinitely large. Therefore k_2 is the point at which $R(k) - L(k)$ is the most likely to be negative.

Figure 2 depicts this case.

Insert figure 2 here.

The equilibrium exists if and only if:

$$R(k_2) \leq L(k_2) \tag{33}$$

It is easy to see that $k_2 = k_0$.

Case 3. $\alpha = 0.5$. $R(k) = \frac{(1-\alpha)\alpha\beta A^2}{1+\beta+\delta} + g + k$, and $R'(k) = 1$, i.e., $R(k)$ is a straight line. $R(k) > L(k)$ for small values of k, and the difference falls until $k = k_0$, where k_0 is the solution of: $L'(k) = 1$, or

$$k_0 = \left[\frac{(\alpha+\beta+\delta)A}{2(1+\beta+\delta)}\right]^2 \tag{34}$$

and rises for $k > k_0$. Hence, the solution of (27) exists if and only if $L(k_0) \geq R(k_0)$.

The existence of solution of the equation $L(k) = R(k)$ is a necessary but not sufficient condition for the existence of equilibrium. Another condition is that $R(k) > \frac{1}{\rho}$, where k solves (27). Define

$$\tilde{k} \equiv (A\alpha\rho)^{1-\alpha} \tag{35}$$

$\tilde{k}$ is defined in such a way that $R^k(\tilde{k}) = \frac{1}{\rho}$. Hence the condition $R^k > \frac{1}{\rho}$ is equivalent to:

$$k < \tilde{k} \tag{36}$$

The last inequality is satisfied, if either $k_0 < \tilde{k}$, or $k_0 \geq \tilde{k}$, but $L(\tilde{k}) \geq R(\tilde{k})$.

Therefore the following is the necessary and sufficient condition for the existence of a steady-state equilibrium described by the system (20)-(26):

$$\left\{ \begin{array}{rl} L(k_0) \geq & R(k_0) \\ \alpha^2(\alpha+\beta+\delta)^2 - 4(1+\beta+\delta)(1-\alpha)\alpha\beta(2\alpha-1) \geq & 0 \\ k_0 < \tilde{k}, \quad \text{or} \quad L(\tilde{k}) > & R(\tilde{k}) \end{array} \right. \tag{37}$$

where $\tilde{k} = \left(\frac{A\alpha}{R^m}\right)^{1-\alpha}$ and

$$\left[\begin{array}{lll} k_0 = & \left(\frac{-\alpha(\alpha+\beta+\delta)+\sqrt{\alpha^2(\alpha+\beta+\delta)^2+4(1+\beta+\delta)(1-\alpha)\alpha\beta(1-2\alpha)}}{2(1-\alpha)\alpha\beta(1-2\alpha)A}\right)^{1/(\alpha-1)} & \text{for} \quad \alpha \neq 0.5 \\ k_0 = & \left[\frac{(\alpha+\beta+\delta)A}{2(1+\beta+\delta)}\right]^2 & \text{for} \quad \alpha = 0.5 \end{array} \right. \tag{38}$$

Figure 1

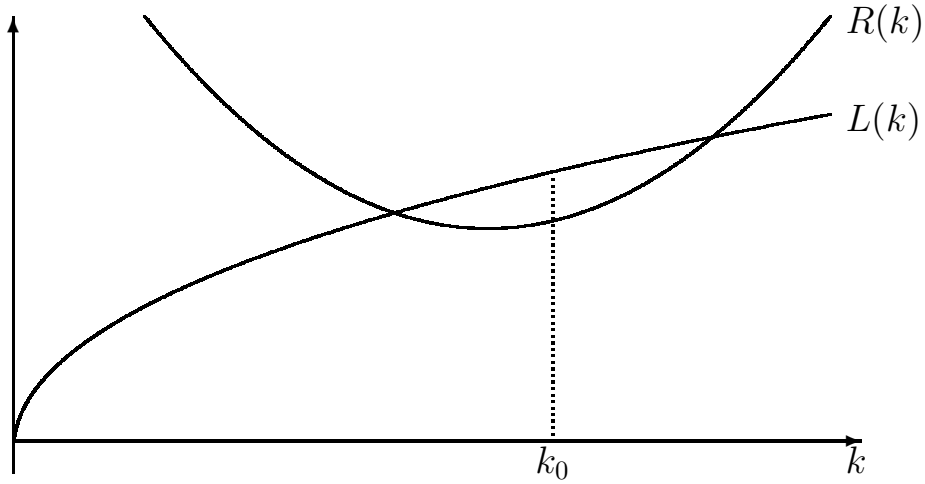
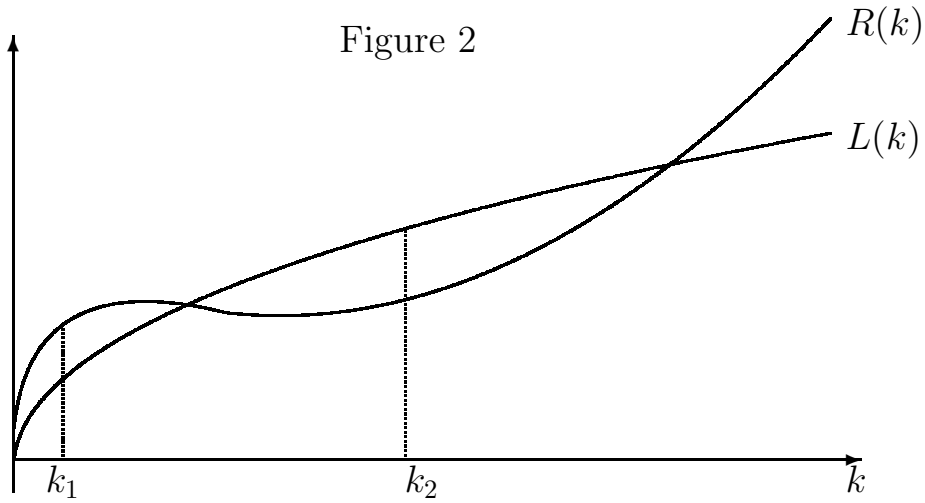


Figure 2



B. Numerical Analysis of the Existence of Equilibrium

Parameter A is a scale parameter, and its choice determines the unit of measurement of output and capital stock. For simplicity we assume $A = 1$. β is typically assumed to be slightly less than 1 for a period of one year. Auerbach and Kotlikoff (1987) assume $\beta = 0.985$. Real business cycle models usually take $\beta = 0.96$. In our model one period is approximately 30 years, and we assume $\beta = 0.6$, which corresponds to the annual rate 0.983.

The value of parameter α, which determines the capital share of output, is set at 0.26, which is slightly higher than 0.25 used by both Auerbach and Kotlikoff (1987) and Bullard and Russell (1999) but lower than is normally used in RBC models.

We assume $\frac{g}{y} = 0.04$. The variable g has a dual interpretation in the model. On the one hand, it is government consumption (typically, 15 - 25 % of GDP in most countries). On the other hand, it is the primary budget deficit, which is often negative, or, if positive, very small. No country has been running a primary budget deficit exceeding 4 % of GDP for a long time (recall, that $\frac{g}{y}$ refers to the *steady state*).

For θ, the utility-from-consumption parameter, we take two values: $\theta = 1$, and $\theta = 1.5$. For θ_m, the utility-from-money parameter, we take three values: $\theta_m = 0.7$, $\theta_m = 1$, and $\theta_m = 1.5$. Given that we are able to analyze analytically existence for the case $\theta = \theta_m = 1$, it is important to take values greater than and less than unity for the numerical analysis. However, θ corresponds to the coefficient of relative risk aversion when utility depends on consumption only. Given that virtually all empirical estimates of this coefficient using the consumption-only utility functions are greater than or equal to unity (See Auerbach and Kotlikoff, 1987, p. 50), we do not consider a value of θ less than one.

Table 2 summarizes the numerical experiments.

Table 2. Existence of Equilibrium

	$\theta_m = 0.7$	$\theta_m = 1$	$\theta_m = 1.5$
$\theta = 1$	$0.17 \leq \delta \leq 1.42$	$0.10 \leq \delta \leq 0.67$	$0.02 \leq \delta \leq 1.20$
$\theta = 1.5$	$0.28 \leq \delta \leq 1.70$	$0.10 \leq \delta \leq 0.86$	$0.04 \leq \delta \leq 0.25$

We see from the table that for all combinations of θ and θ_m there exists a reasonably wide range of values of δ compatible with the real solution to the system (11)-(19).

C. PROOFS

Proof of Proposition 1

Substituting out s and w, replacing the second-period budget constraint with the goods market clearing condition [29], and taking into account that $R^m = 1/\rho$, the system (11)-(19) can be rewritten as:

$$
\begin{aligned}
Ak^{\alpha} &= c_1 + c_2 + g + k && (39)\\
R &= A\alpha k^{\alpha-1} && (40)\\
c_2 &= (\beta R)^{1/\theta} c_1 && (41)\\
R &= \frac{1}{\rho} + \frac{\delta c_2^{\theta}}{\beta m^{\theta_m}} && (42)\\
c_1 &= A(1-\alpha)k^{\alpha} - k - b - m && (43)\\
g &= (1 - \frac{1}{\rho})m + b(1-R) && (44)
\end{aligned}
$$

Taking into account (41), equation (43) can be written as:

$$c_2(\beta R)^{-\frac{1}{\theta}} = A(1-\alpha)k^{\alpha} - k - b - m \qquad (45)$$

Analysis of the competitive equilibrium proceeds in the following way. We search for conditions under which the system (39) - (42), (44), (45) can be decomposed into the "real block" (equations (39) - (41)), and the "monetary block" [equations (42), (44), (45)], so that a change in ρ affects only the monetary block variables, but not the real block ones (especially, k and R), and hence superneutrality holds.

The monetary block can be considered a system of three equations with three unknowns: c_2, m, and b, which are functions of k, R and the exogenous parameters.

It is easy to see that among the three endogenous variables of the monetary block only c_2 is present in (39) - (41). Therefore as long as c_2 is not affected by a change in ρ *within the monetary block*, i.e. $\frac{\partial c_2}{\partial \rho} = 0$, then superneutrality holds, i.e. a change in ρ does not affect c_1, k, R and y.

[29] Of the four constraints (the two private agent's budget constraints, the government budget constraint, and the goods market clearing condition) any one of them is redundant. It can be written as a linear combination of the others.

Totally differentiating the monetary block equations we get:

$$\frac{d\rho}{\rho(\rho R-1)} = -\frac{\theta_m}{m}dm + \frac{\theta}{c_2}dc_2 \tag{46}$$

$$(\beta R)^{-\frac{1}{\theta}}dc_2 = -db - dm \tag{47}$$

$$0 = dm - \frac{m}{\rho^2}d\rho - \frac{1}{\rho}dm + db(1-R) \tag{48}$$

Successive elimination of db and dm yields:

$$\left[(R-1)(\beta R)^{-\frac{1}{\theta}}\frac{\theta_m}{m} + \frac{\theta(R-\frac{1}{\rho})}{c_2}\right]dc_2 = \frac{1-\theta_m}{\rho^2}d\rho \tag{49}$$

As $(R-1)(\beta R)^{-\frac{1}{\theta}}\frac{\theta_m}{m} + \frac{\theta(R-R^m)}{c_2} > 0$, $\frac{\partial c_2}{\partial \rho} = 0$ if and only if $\theta_m = 1$.

Proof of Lemma 1

If R does not change, the values of the capital stock and output do not change either. Totally differentiating system (39)-(44) under this assumption yields:

$$dc_1 + dc_2 + dG = 0 \tag{50}$$

$$dc_2 = (\beta R)^{1/\theta}dc_1 \tag{51}$$

$$\frac{d\rho}{\rho(\rho R-1)} = -\frac{\theta_m}{m}dm + \frac{\theta}{c_2}dc_2 \tag{52}$$

$$dc_1 = -db - dm \tag{53}$$

$$dG = dm + \frac{m}{\rho^2}d\rho - \frac{1}{\rho}dm + db(1-R) \tag{54}$$

Successive elimination of dc_1, dc_2, dm, and db yields:

$$-\frac{1-\theta_m}{\rho^2}d\rho = dG\left[\frac{\Pi+R}{1+\Pi}\frac{\theta_m}{m} + \frac{\theta}{c_2}\frac{\Pi}{1+\Pi}(R-\frac{1}{\rho})\right], \tag{55}$$

where $\Pi = (\beta R)^{1/\theta}$. The expression in square brackets is always positive. Hence the sign of $\frac{\partial G}{\partial \rho}$ is the opposite of the sign of $1-\theta_m$. Therefore, $\frac{\partial g}{\partial \rho}$ is greater (less) than zero if and only if θ_m is greater (less) than 1. Q.E.D.

Proof of Lemma 2

Let $R^m \equiv \frac{1}{\rho}, \Pi \equiv (R\beta)^{1/\theta}$ and $Z \equiv \left(\frac{R\,\delta}{R-R^m}\right)^{1/\theta}$. Budget constraints (3) and (4) can be combined to form a single intertemporal budget constraint:

$$c_1 + \frac{c_2}{R} = w - m\left(1 - \frac{R^m}{R}\right) \tag{56}$$

The maximization of the utility function:

$$U(c_1, c_2, m) = \frac{c_1^{1-\theta}}{1-\theta} + \beta\frac{c_2^{1-\theta}}{1-\theta} + \delta\frac{m^{1-\theta}}{1-\theta}$$

subject to the intertemporal constraint of the household yields the following household demand functions:

$$c_1 = w\frac{R}{R + \Pi + (R - R^m)Z}$$

$$m = Zc_1 = w\frac{RZ}{R + \Pi + (R - R^m)Z}$$

and we have

$$c_1 + m = (1 + Z)c_1 = w\frac{R\,(1 + Z)}{R + \Pi + (R - R^m)Z}$$

The residual demand for government bonds is

$$\begin{aligned} b &= w - c_1 - m - k \\ &= w\,\frac{R + \Pi + (R - R^m)\,Z - R\,(1 + Z)}{R + \Pi + (R - R^m)\,Z} - k \end{aligned}$$

Hence,

$$b = w\left[\frac{\Pi - R^m\,Z}{R + \Pi + (R - R^m)\,Z}\right] - k \tag{57}$$

The equilibrium government revenue function is

$$\begin{aligned} G(R, R^m) &= (1 - R^m)m + (1 - R)b \\ &= (1 - R^m)w\frac{RZ}{R + \Pi + (R - R^m)Z} + (1 - R)\left\{w\left[\frac{\Pi - R^m Z}{R + \Pi + (R - R^m)Z}\right] - k\right\} \\ &= \frac{w}{R + \Pi + (R - R^m)Z}\left[(1 - R^m)\,R\,Z + (1 - R)\,(\Pi - R^m Z)\right] - (1 - R)k \end{aligned}$$

Therefore,

$$G(R, R^m) = (R - 1)k + w\left[\frac{(R - R^m)Z - (R - 1)\Pi}{(R - R^m)Z + (R + \Pi)}\right] \tag{58}$$

or

$$G(R, R^m) = (R-1)\left[k - w\frac{\Pi}{(R-R^m)Z + (R+\Pi)}\right] + w\frac{(R-R^m)Z}{(R-R^m)Z + (R+\Pi)} \tag{59}$$

Define $\Omega \equiv (R-R^m)Z + (R+\Pi)$. Then we have

$$G > 0 \quad \text{if and only if} \quad (R-1)\left[1 - \frac{w}{k}\frac{\Pi}{\Omega}\right] + \frac{w}{k}\frac{(R-R^m)\,Z}{\Omega} > 0 \tag{60}$$

The rest of the proof will proceed as follows. First, we will prove that

$$\frac{[(R-1) + (R-R^m)Z]\,[(\Pi - R^m)Z) - R\Pi'] + (R-R^m)\,[RZ'(1+\Pi)]}{[(R-R^m)Z + (R+\Pi)]^2} < 0 \tag{61}$$

Second we will show that given (61),

$$(R-1)\left\{1 - \frac{w'}{k'}\frac{\Pi}{\Omega}\right\} + \frac{w'}{k'}\frac{(R-R^m)Z}{\Omega} > 0 \tag{62}$$

is a sufficient condition for $\frac{\partial G}{\partial R} < 0$. Third, we will prove that $G > 0$ implies (62).

Part 1.

Since $Z' < 0$ and $k' < 0$, a sufficient condition for (61) is:

$$\Pi - R^m Z < R\Pi'$$

which is

$$(R\,\beta)^{1/\theta}\,(1 - \frac{1}{\theta}) < R^m\left(\frac{R\delta}{R-R^m}\right)^{1/\theta}$$

If $\theta \leq 1$ then this condition always holds.

Part 2.

Differentiating $G(R, R^m) = (1-R^m)m + (1-R)b$ with respect to R, we get:

$$\frac{\partial g}{\partial R} = (1-R^m)\frac{\partial m}{\partial R} + (1-R)\frac{\partial b}{\partial R} - b \tag{63}$$

Given that $b > 0$, $(1-R^m)\frac{\partial m}{\partial R} + (1-R)\frac{\partial b}{\partial R} < 0$ is sufficient for $\frac{\partial G}{\partial R} < 0$. On the other hand, differentiating (58) with respect to R, we get:

$$\frac{\partial G}{\partial R} = k + (R-1)k' + w'\left[\frac{(R-R^m)Z-(R-1)\Pi}{(R-R^m)Z+(R+\Pi)}\right] + \tag{64}$$

$$\frac{w}{\Omega^2}\left\{\left[Z+(R-R^m)Z'-\Pi-(R-1)\Pi'\right]\left[(R-R^m)\,Z+(R+\Pi)\right]\right\} -$$

$$\frac{w}{\Omega^2}\left\{\left[Z+(R-R^m)Z'+1+\Pi'\right]\left[(R-R^m)Z-(R-1)\Pi\right]\right\}$$

In addition, (57) can be rewritten as:

$$b = w\left\{\frac{\left[(R-R^m)Z+\Pi+R\right]\left[\Pi-R^mZ\right]}{\left[(R-R^m)Z+\Pi+R\right]^2}\right\} - k \tag{65}$$

Combining (63), (64) and (65) then gives us

$$\begin{aligned}
&(1-R^m)\frac{\partial m}{\partial R} + (1-R))\frac{\partial b}{\partial R} = \frac{\partial G}{\partial R} + b = \\
= \quad & (R-1)k' + w'\left[\frac{(R-R^m)Z-(R-1)\Pi}{(R-R^m)Z+(R+\Pi)}\right] \\
& + w\frac{\left[(R-R^m)Z+(R-1)\right]\left[(\Pi-R^mZ)-R\Pi'\right]+RZ'(R-R^m)\,(1+\Pi)}{\left[(R-R^m)\,Z+(R+\Pi)\right]^2}
\end{aligned}$$

Therefore, $(1-R^m)\frac{\partial G}{\partial R} + (1-R) < 0$ can be written as

$$\begin{aligned}
&(R-1)\left\{k' - w'\frac{\Pi}{(R-R^m)\,Z+(R+\Pi)} + w\,\frac{(\Pi-R^m\,Z)-R\,\Pi'}{\left[(R-R^m)\,Z+(R+\Pi)\right]^2}\right\} \\
< \quad & -w'\,\frac{(R-R^m)\,Z}{(R-R^m)\,Z+(R+\Pi)} - w\,\frac{(R-R^m)\,\left\{Z\,\left[(\Pi-R^m\,Z)-R\,\Pi'\right]+R\,Z'\,(1+\Pi)\right\}}{\left[(R-R^m)\,Z+(R+\Pi)\right]^2}\,.
\end{aligned}$$

We can rewrite this condition as

$$\begin{aligned}
&(R-1)\left\{k' - w'\frac{\Pi}{(R-R^m)\,Z+(R+\Pi)}\right\} + w'\frac{(R-R^m)Z}{(R-R^m)Z+(R+\Pi)} \\
< \quad & -w\frac{\left[(R-1)+(R-R^m)\,Z\right]\,\left[(\Pi-R^m\,Z)-R\Pi'\right]+(R-R^m)\,\left[R\,Z'\,(1+\Pi)\right]}{\left[(R-R^m)\,Z+(R+\Pi)\right]^2}\,.
\end{aligned}$$

Given the result proved in Part 1,

$$(R-1)\left\{k' - w'\frac{\Pi}{(R-R^m)Z+(R+\Pi)}\right\} + w'\frac{(R-R^m)Z}{(R-R^m)Z+(R+\Pi)} < 0 \tag{66}$$

is a sufficient condition for $\frac{\partial G}{\partial R} < 0$.

Dividing (66) though by k' and using the definition of Ω, the last inequality can be written as:

$$(R-1)\left\{1 - \frac{w'}{k'}\frac{\Pi}{\Omega}\right\} + \frac{w'}{k'}\frac{(R-R^m)Z}{\Omega} > 0 \tag{67}$$

Part 3.

We will show that

$$(R-1)\left[1-\frac{w}{k}\frac{\Pi}{\Omega}\right]+\frac{w}{k}\frac{(R-R^m)\,Z}{\Omega}\geq 0 \tag{68}$$

(which is true by assumption $G>0$) implies

$$(R-1)\left\{1-\frac{w'}{k'}\frac{\Pi}{\Omega}\right\}+\frac{w'}{k'}\frac{(R-R^m)\,Z}{\Omega}>0 \tag{69}$$

We can rewrite condition (68) as

$$(R-1)+\frac{1}{\Omega}\frac{w}{k}\left[(R-R^m)Z-\Pi\right]\geq 0 \tag{70}$$

We can rewrite condition (69) as

$$(R-1)+\frac{1}{\Omega}\frac{w'}{k'}\left[(R-R^m)Z-\Pi\right]>0 \tag{71}$$

It is easy to verify that

$$\frac{w'}{k'}=R\,(1-\alpha)$$

and

$$\frac{w}{k}=\frac{(1-\alpha)\left(\frac{R}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}}{\left(\frac{R}{\alpha}\right)^{\frac{1}{\alpha-1}}}=\frac{R\,(1-\alpha)}{\alpha}\,.$$

So

$$\frac{w'}{k'}=\alpha\frac{w}{k}\,.$$

Thus, condition (71) can be rewritten as

$$(R-1)+\frac{1}{\Omega}\alpha\frac{w}{k}\left[(R-R^m)Z-\Pi\right]\geq 0 \tag{72}$$

The LHS of condition (70) is the sum of the positive term $(R-1)$ and the term with an ambiguous sign, $\frac{1}{\Omega}\frac{w}{k}\left[(R-R^m)Z-\Pi\right]$. The LHS of (72) is the sum of the same positive term $(R-1)$ and the same ambiguously signed term multiplied by a constant $\alpha<1$. If this ambiguously signed term is positive, condition (72) holds trivially. If this term is negative, then the LHS of condition (72) is greater than the LHS of (70). In the latter case condition (72) must hold as long as the first condition holds. Q.E.D.

Proof of Lemma 3:

The proof follows immediately from Lemma 1. Totally differentiating the definition of the government revenue function, one obtains:

$$dG = \frac{\partial G}{\partial \rho} d\rho + \frac{\partial G}{\partial R} dR$$

Therefore, if the government revenue remains constant, then

$$\frac{\partial R}{\partial \rho} = -\frac{\frac{\partial G}{\partial \rho}}{\frac{\partial G}{\partial R}}$$

Given that the production function is Cobb-Douglas, and

$$k = \left[\frac{A\alpha}{R}\right]^{\frac{1}{1-\alpha}},$$

$$\text{sign}\left[\frac{\partial k}{\partial \rho}\right] = \text{sign}\left[\frac{\frac{\partial G}{\partial \rho}}{\frac{\partial G}{\partial R}}\right]$$

Taking into account the result of Lemma 2,

$$\text{sign}\left[\frac{\partial k}{\partial \rho}\right] = -\text{sign}\left[\frac{\partial G}{\partial \rho}\right]$$

Therefore, by Lemma 1, $\frac{\partial k}{\partial \rho} > 0$ for $\theta_m < 1$; $\frac{\partial k}{\partial \rho} < 0$ for $\theta_m > 1$. Q.E.D.

Proof of Proposition 2:

The proof follows directly from Lemmas 2 and 3. If conditions of Lemma 2 are satisfied, i.e., if $b > 0$ and $R > 1$, then by continuity, there exists $\bar{\theta}$ such that $\frac{\partial G}{\partial R} < 0$ for any $\theta_m \in (1, \bar{\theta})$. Therefore, by Lemma 3, $\frac{\partial k}{\partial \rho} > 0$ for $\theta < 1$, and $\frac{\partial k}{\partial \rho} < 0$ for $\theta > 1$. Q.E.D.