Notes on Macroeconomic Theory

Steve Williamson
Dept. of Economics
Washington University in St. Louis
St. Louis, MO 63130

September 2006
Chapter 1

Simple Representative Agent Models

This chapter deals with the simplest kind of macroeconomic model, which abstracts from all issues of heterogeneity and distribution among economic agents. Here, we study an economy consisting of a representative firm and a representative consumer. As we will show, this is equivalent, under some circumstances, to studying an economy with many identical firms and many identical consumers. Here, as in all the models we will study, economic agents optimize, i.e. they maximize some objective subject to the constraints they face. The preferences of consumers, the technology available to firms, and the endowments of resources available to consumers and firms, combined with optimizing behavior and some notion of equilibrium, allow us to use the model to make predictions. Here, the equilibrium concept we will use is competitive equilibrium, i.e. all economic agents are assumed to be price-takers.

1.1 A Static Model

1.1.1 Preferences, endowments, and technology

There is one period and $N$ consumers, who each have preferences given by the utility function $u(c, \ell)$, where $c$ is consumption and $\ell$ is leisure. Here, $u(\cdot, \cdot)$ is strictly increasing in each argument, strictly concave, and
CHAPTER 1. SIMPLE REPRESENTATIVE AGENT MODELS

twice differentiable. Also, assume that \( \lim_{c \to 0} u_1(c, \ell) = \infty, \ell > 0 \), and \( \lim_{c \to 0} u_2(c, \ell) = \infty, c > 0 \). Here, \( u_i(c, \ell) \) is the partial derivative with respect to argument \( i \) of \( u(c, \ell) \). Each consumer is endowed with one unit of time, which can be allocated between work and leisure. Each consumer also owns \( \frac{k_0}{N} \) units of capital, which can be rented to firms.

There are \( M \) firms, which each have a technology for producing consumption goods according to

\[
y = zf(k, n),
\]

where \( y \) is output, \( k \) is the capital input, \( n \) is the labor input, and \( z \) is a parameter representing total factor productivity. Here, the function \( f(\cdot, \cdot) \) is strictly increasing in both arguments, strictly quasiconcave, twice differentiable, and homogeneous of degree one. That is, production is constant returns to scale, so that

\[
\lambda y = zf(\lambda k, \lambda n),
\]

for \( \lambda > 0 \). Also, assume that \( \lim_{k \to 0} f_1(k, n) = \infty, \lim_{k \to \infty} f_1(k, n) = 0, \lim_{n \to 0} f_2(k, n) = \infty, \) and \( \lim_{n \to \infty} f_2(k, n) = 0 \).

1.1.2 Optimization

In a competitive equilibrium, we can at most determine all relative prices, so the price of one good can arbitrarily be set to 1 with no loss of generality. We call this good the numeraire. We will follow convention here by treating the consumption good as the numeraire. There are markets in three objects, consumption, leisure, and the rental services of capital. The price of leisure in units of consumption is \( w \), and the rental rate on capital (again, in units of consumption) is \( r \).

Consumer’s Problem

Each consumer treats \( w \) as being fixed, and maximizes utility subject to his/her constraints. That is, each solves

\[
\max_{c, \ell, k_0} u(c, \ell)
\]
subject to
\[ c \leq w(1 - \ell) + rk_s \]  \hspace{1cm} (1.2)
\[ 0 \leq k_s \leq \frac{k_0}{N} \]  \hspace{1cm} (1.3)
\[ 0 \leq \ell \leq 1 \]  \hspace{1cm} (1.4)
\[ c \geq 0 \]  \hspace{1cm} (1.5)

Here, \( k_s \) is the quantity of capital that the consumer rents to firms, (1.2) is the budget constraint, (1.3) states that the quantity of capital rented must be positive and cannot exceed what the consumer is endowed with, (1.4) is a similar condition for leisure, and (1.5) is a nonnegativity constraint on consumption.

Now, given that utility is increasing in consumption (more is preferred to less), we must have \( k_s = \frac{k_0}{N} \), and (1.2) will hold with equality. Our restrictions on the utility function assure that the nonnegativity constraints on consumption and leisure will not be binding, and in equilibrium we will never have \( \ell = 1 \), as then nothing would be produced, so we can safely ignore this case. The optimization problem for the consumer is therefore much simplified, and we can write down the following Lagrangian for the problem.

\[ \mathcal{L} = u(c, \ell) + \mu(w + r \frac{k_0}{N} - w\ell - c), \]

where \( \mu \) is a Lagrange multiplier. Our restrictions on the utility function assure that there is a unique optimum which is characterized by the following first-order conditions.

\[ \frac{\partial \mathcal{L}}{\partial c} = u_1 - \mu = 0 \]
\[ \frac{\partial \mathcal{L}}{\partial \ell} = u_2 - \mu w = 0 \]
\[ \frac{\partial \mathcal{L}}{\partial \mu} = w + r \frac{k_0}{N} - w\ell - c = 0 \]

Here, \( u_i \) is the partial derivative of \( u(\cdot, \cdot) \) with respect to argument \( i \). The above first-order conditions can be used to solve out for \( \mu \) and \( c \) to obtain

\[ wu_1(w + r \frac{k_0}{N} - w\ell, \ell) - u_2(w + r \frac{k_0}{N} - w\ell, \ell) = 0, \]  \hspace{1cm} (1.6)
which solves for the desired quantity of leisure, \( \ell \), in terms of \( w, r, \) and \( \frac{k_0}{N} \). Equation (1.6) can be rewritten as

\[
\frac{u_2}{u_1} = w,
\]

i.e. the marginal rate of substitution of leisure for consumption equals the wage rate. Diagrammatically, in Figure 1.1, the consumer’s budget constraint is ABD, and he/she maximizes utility at E, where the budget constraint, which has slope \(-w\), is tangent to the highest indifference curve, where an indifference curve has slope \(-\frac{u_2}{u_1}\).

**Firm’s Problem**

Each firm chooses inputs of labor and capital to maximize profits, treating \( w \) and \( r \) as being fixed. That is, a firm solves

\[
\max_{k,n} [zf(k, n) - rk - wn],
\]
and the first-order conditions for an optimum are the marginal product conditions

\[ zf_1 = r, \]
\[ zf_2 = w, \]

(1.7) (1.8)

where \( f_i \) denotes the partial derivative of \( f(\cdot, \cdot) \) with respect to argument \( i \). Now, given that the function \( f(\cdot, \cdot) \) is homogeneous of degree one, Euler’s law holds. That is, differentiating (1.1) with respect to \( \lambda \), and setting \( \lambda = 1 \), we get

\[ zf(k, n) = zf_1 k + zf_2 n. \]

(1.9)

Equations (1.7), (1.8), and (1.9) then imply that maximized profits equal zero. This has two important consequences. The first is that we do not need to be concerned with how the firm’s profits are distributed (through shares owned by consumers, for example). Secondly, suppose \( k^* \) and \( n^* \) are optimal choices for the factor inputs, then we must have

\[ zf(k, n) - rk - wn = 0 \]  

(1.10)

for \( k = k^* \) and \( n = n^* \). But, since (1.10) also holds for \( k = \lambda k^* \) and \( n = \lambda n^* \) for any \( \lambda > 0 \), due to the constant returns to scale assumption, the optimal scale of operation of the firm is indeterminate. It therefore makes no difference for our analysis to simply consider the case \( M = 1 \) (a single, representative firm), as the number of firms will be irrelevant for determining the competitive equilibrium.

### 1.1.3 Competitive Equilibrium

A competitive equilibrium is a set of quantities, \( c, \ell, n, k, \) and prices \( w \) and \( r \), which satisfy the following properties.

1. Each consumer chooses \( c \) and \( \ell \) optimally given \( w \) and \( r \).
2. The representative firm chooses \( n \) and \( k \) optimally given \( w \) and \( r \).
CHAPTER 1. SIMPLE REPRESENTATIVE AGENT MODELS

Here, there are three markets: the labor market, the market for consumption goods, and the market for rental services of capital. In a competitive equilibrium, given (3), the following conditions then hold.

\[ N(1 - \ell) = n \]  
(1.11)

\[ y = Nc \]  
(1.12)

\[ k_0 = k \]  
(1.13)

That is, supply equals demand in each market given prices. Now, the total value of excess demand across markets is

\[ Nc - y + w[n - N(1 - \ell)] + r(k - k_0), \]

but from the consumer’s budget constraint, and the fact that profit maximization implies zero profits, we have

\[ Nc - y + w[n - N(1 - \ell)] + r(k - k_0) = 0. \]  
(1.14)

Note that (1.14) would hold even if profits were not zero, and were distributed lump-sum to consumers. But now, if any 2 of (1.11), (1.12), and (1.13) hold, then (1.14) implies that the third market-clearing condition holds. Equation (1.14) is simply Walras’ law for this model. Walras’ law states that the value of excess demand across markets is always zero, and this then implies that, if there are \( M \) markets and \( M - 1 \) of those markets are in equilibrium, then the additional market is also in equilibrium. We can therefore drop one market-clearing condition in determining competitive equilibrium prices and quantities. Here, we eliminate (1.12).

The competitive equilibrium is then the solution to (1.6), (1.7), (1.8), (1.11), and (1.13). These are five equations in the five unknowns \( \ell, n, k, w, \) and \( r, \) and we can solve for \( c \) using the consumer’s budget constraint. It should be apparent here that the number of consumers, \( N, \) is virtually irrelevant to the equilibrium solution, so for convenience we can set \( N = 1, \) and simply analyze an economy with a single representative consumer. Competitive equilibrium might seem inappropriate when there is one consumer and one firm, but as we have shown, in this context our results would not be any different if there were many firms.
1.1. A STATIC MODEL

and many consumers. We can substitute in equation (1.6) to obtain an equation which solves for equilibrium $\ell$.

$$zf_2(k_0, 1 - \ell)u_1(zf(k_0, 1 - \ell), \ell) - u_2(zf(k_0, 1 - \ell), \ell) = 0 \quad (1.15)$$

Given the solution for $\ell$, we then substitute in the following equations to obtain solutions for $r, w, n, k, a n d c$.

$$zf_1(k_0, 1 - \ell) = r \quad (1.16)$$

$$zf_2(k_0, 1 - \ell) = w \quad (1.17)$$

$$n = 1 - \ell$$

$$k = k_0$$

$$c = zf(k_0, 1 - \ell) \quad (1.18)$$

It is not immediately apparent that the competitive equilibrium exists and is unique, but we will show this later.

1.1.4 Pareto Optimality

A Pareto optimum, generally, is defined to be some allocation (an allocation being a production plan and a distribution of goods across economic agents) such that there is no other allocation which some agents strictly prefer which does not make any agents worse off. Here, since we have a single agent, we do not have to worry about the allocation of goods across agents. It helps to think in terms of a fictitious social planner who can dictate inputs to production by the representative firm, can force the consumer to supply the appropriate quantity of labor, and then distributes consumption goods to the consumer, all in a way that makes the consumer as well off as possible. The social planner determines a Pareto optimum by solving the following problem.

$$\max_{c, \ell} u(c, \ell)$$

subject to

$$c = zf(k_0, 1 - \ell) \quad (1.19)$$
CHAPTER 1. SIMPLE REPRESENTATIVE AGENT MODELS

Given the restrictions on the utility function, we can simply substitute using the constraint in the objective function, and differentiate with respect to $\ell$ to obtain the following first-order condition for an optimum.

$$zf_2(k_0, 1 - \ell)u_1[zf(k_0, 1 - \ell), \ell] - u_2[zf(k_0, 1 - \ell), \ell] = 0 \quad (1.20)$$

Note that (1.15) and (1.20) are identical, and the solution we get for $c$ from the social planner’s problem by substituting in the constraint will yield the same solution as from (1.18). That is, the competitive equilibrium and the Pareto optimum are identical here. Further, since $u(\cdot, \cdot)$ is strictly concave and $f(\cdot, \cdot)$ is strictly quasiconcave, there is a unique Pareto optimum, and the competitive equilibrium is also unique.

Note that we can rewrite (1.20) as

$$zf_2 = \frac{u_2}{u_1},$$

where the left side of the equation is the marginal rate of transformation, and the right side is the marginal rate of substitution of consumption for leisure. In Figure 1.2, AB is equation (1.19) and the Pareto optimum is at D, where the highest indifference curve is tangent to the production possibilities frontier. In a competitive equilibrium, the representative consumer faces budget constraint EFB and maximizes at point D where the slope of the budget line, $-w$, is equal to $-\frac{u_2}{u_1}$.

In more general settings, it is true under some restrictions that the following hold.

1. A competitive equilibrium is Pareto optimal (First Welfare Theorem).

2. Any Pareto optimum can be supported as a competitive equilibrium with an appropriate choice of endowments. (Second Welfare Theorem).

The non-technical assumptions required for (1) and (2) to go through include the absence of externalities, completeness of markets, and absence of distorting taxes (e.g. income taxes and sales taxes). The First Welfare Theorem is quite powerful, and the general idea goes back as far as Adam Smith’s Wealth of Nations. In macroeconomics, if we can
1.1. A STATIC MODEL

Figure 1.2: Pareto Optimum and Competitive Equilibrium

![Graph showing consumption and leisure]

Figure 1.2:

successfully explain particular phenomena (e.g. business cycles) using a competitive equilibrium model in which the First Welfare Theorem holds, we can then argue that the existence of such phenomena is not grounds for government intervention.

In addition to policy implications, the equivalence of competitive equilibria and Pareto optima in representative agent models is useful for computational purposes. That is, it can be much easier to obtain competitive equilibria by first solving the social planner’s problem to obtain competitive equilibrium quantities, and then solving for prices, rather than solving simultaneously for prices and quantities using market-clearing conditions. For example, in the above example, a competitive equilibrium could be obtained by first solving for $c$ and $\ell$ from the social planner’s problem, and then finding $w$ and $r$ from the appropriate marginal conditions, (1.16) and (1.17). Using this approach does not make much difference here, but in computing numerical solutions in dynamic models it can make a huge difference in the computational burden.
1.1.5 Example

Consider the following specific functional forms. For the utility function, we use
\[ u(c, \ell) = \frac{c^{1-\gamma} - 1}{1-\gamma} + \ell, \]
where \( \gamma > 0 \) measures the degree of curvature in the utility function with respect to consumption (this is a “constant relative risk aversion” utility function). Note that
\[ \lim_{\gamma \to 1} \frac{c^{1-\gamma} - 1}{1-\gamma} = \lim_{\gamma \to 1} \frac{d}{d\gamma} \left[ c^{(1-\gamma)\log c - 1} \right] = \log c, \]
using L’Hospital’s Rule. For the production technology, use
\[ f(k, n) = k^\alpha n^{1-\alpha}, \]
where \( 0 < \alpha < 1 \). That is, the production function is Cobb-Douglas.

The social planner’s problem here is then
\[ \max_{\ell} \left\{ \frac{[zk_0^\alpha (1-\ell)1-\alpha]^{1-\gamma} - 1}{1-\gamma} + \ell \right\}, \]
and the solution to this problem is
\[ \ell = 1 - [((1-\alpha)(zk_0^\alpha)^{1-\gamma}]^{1/(1-\gamma)}, \tag{1.21} \]
As in the general case above, this is also the competitive equilibrium solution. Solving for \( c \), from (1.19), we get
\[ c = [((1-\alpha)(zk_0^\alpha]^1/(1-\gamma)] \tag{1.22} \]
and from (1.17), we have
\[ w = [((1-\alpha)(zk_0^\alpha)^{\gamma/(1-\gamma)]} \tag{1.23} \]
From (1.22) and (1.23) clearly \( c \) and \( w \) are increasing in \( z \) and \( k_0 \). That is, increases in productivity and in the capital stock increase aggregate consumption and real wages. However, from equation (1.21) the effects
on the quantity of leisure (and therefore on employment) are ambiguous. Which way the effect goes depends on whether $\gamma < 1$ or $\gamma > 1$. With $\gamma < 1$, an increase in $z$ or in $k_0$ will result in a decrease in leisure, and an increase in employment, but the effects are just the opposite if $\gamma > 1$. If we want to treat this as a simple model of the business cycle, where fluctuations are driven by technology shocks (changes in $z$), these results are troubling. In the data, aggregate output, aggregate consumption, and aggregate employment are mutually positively correlated. However, this model can deliver the result that employment and output move in opposite directions. Note however, that the real wage will be procyclical (it goes up when output goes up), as is the case in the data.

1.1.6 Linear Technology - Comparative Statics

This section illustrates the use of comparative statics, and shows, in a somewhat more general sense than the above example, why a productivity shock might give a decrease or an increase in employment. To make things clearer, we consider a simplified technology, 

$$y = zn,$$

i.e. we eliminate capital, but still consider a constant returns to scale technology with labor being the only input. The social planner’s problem for this economy is then

$$\max_{\ell} u[z(1 - \ell), \ell],$$

and the first-order condition for a maximum is

$$-zu_1[z(1 - \ell), \ell] + u_2[z(1 - \ell), \ell] = 0. \tag{1.24}$$

Here, in contrast to the example, we cannot solve explicitly for $\ell$, but note that the equilibrium real wage is

$$w = \frac{\partial y}{\partial n} = z,$$

so that an increase in productivity, $z$, corresponds to an increase in the real wage faced by the consumer. To determine the effect of an increase
in $z$ on $\ell$, apply the implicit function theorem and totally differentiate (1.24) to get

$$
[-u_1 - z(1 - \ell)u_{11} + u_{21}(1 - \ell)]dz 
+ (z^2u_{11} - 2zu_{12} + u_{22})d\ell = 0.
$$

We then have

$$
\frac{d\ell}{dz} = \frac{u_1 + z(1 - \ell)u_{11} - u_{21}(1 - \ell)}{z^2u_{11} - 2zu_{12} + u_{22}}. \tag{1.25}
$$

Now, concavity of the utility function implies that the denominator in (1.25) is negative, but we cannot sign the numerator. In fact, it is easy to construct examples where $\frac{d\ell}{dz} > 0$, and where $\frac{d\ell}{dz} < 0$. The ambiguity here arises from opposing income and substitution effects.

In Figure 1.3, AB denotes the resource constraint faced by the social planner, $c = z_1(1 - \ell)$, and BD is the resource constraint with a higher level of productivity, $z_2 > z_1$. As shown, the social optimum (also the competitive equilibrium) is at E initially, and at F after the increase in productivity, with no change in $\ell$ but higher $c$. Effectively, the representative consumer faces a higher real wage, and his/her response can be decomposed into a substitution effect (E to G) and an income effect (G to F).

Algebraically, we can determine the substitution effect on leisure by changing prices and compensating the consumer to hold utility constant, i.e.

$$
u(c, \ell) = h, \tag{1.26}
$$

where $h$ is a constant, and

$$
-zu_1(c, \ell) + u_2(c, \ell) = 0 \tag{1.27}
$$

Totally differentiating (1.26) and (1.27) with respect to $c$ and $\ell$, and using (1.27) to simplify, we can solve for the substitution effect $\frac{d\ell}{dz}(\text{subst.})$ as follows.

$$
\frac{d\ell}{dz}(\text{subst.}) = \frac{u_1}{z^2u_{11} - 2zu_{12} + u_{22}} < 0.
$$

From (1.25) then, the income effect $\frac{d\ell}{dz}(\text{inc.})$ is just the remainder,

$$
\frac{d\ell}{dz}(\text{inc.}) = \frac{z(1 - \ell)u_{11} - u_{21}(1 - \ell)}{z^2u_{11} - 2zu_{12} + u_{22}} > 0,
$$
provided \( \ell \) is a normal good. Therefore, in order for a model like this one to be consistent with observation, we require a substitution effect that is large relative to the income effect. That is, a productivity shock, which increases the real wage and output, must result in a decrease in leisure in order for employment to be procyclical, as it is in the data. In general, preferences and substitution effects are very important in equilibrium theories of the business cycle, as we will see later.

1.2 Government

So that we can analyze some simple fiscal policy issues, we introduce a government sector into our simple static model in the following manner. The government makes purchases of consumption goods, and finances these purchases through lump-sum taxes on the representative consumer. Let \( q \) be the quantity of government purchases, which is treated as being exogenous, and let \( \tau \) be total taxes. The government
budget must balance, i.e.

\[ g = \tau. \]  

(1.28)

We assume here that the government destroys the goods it purchases. This is clearly unrealistic (in most cases), but it simplifies matters, and does not make much difference for the analysis, unless we wish to consider the optimal determination of government purchases. For example, we could allow government spending to enter the consumer’s utility function in the following way.

\[ w(c, \ell, g) = u(c, \ell) + v(g) \]

Given that utility is separable in this fashion, and \( g \) is exogenous, this would make no difference for the analysis. Given this, we can assume \( v(g) = 0 \).

As in the previous section, labor is the only factor of production, i.e. assume a technology of the form

\[ y = zn. \]

Here, the consumer’s optimization problem is

\[ \max_{c, \ell} u(c, \ell) \]

subject to

\[ c = w(1 - \ell) - \tau, \]

and the first-order condition for an optimum is

\[ -wu_1 + u_2 = 0. \]

The representative firm’s profit maximization problem is

\[ \max_n (z - w)n. \]

Therefore, the firm’s demand for labor is infinitely elastic at \( w = z \).

A competitive equilibrium consists of quantities, \( c, \ell, n, \) and \( \tau, \) and a price, \( w, \) which satisfy the following conditions:

1. The representative consumer chooses \( c \) and \( \ell \) to maximize utility, given \( w \) and \( \tau. \)
2. The representative firm chooses \( n \) to maximize profits, given \( w \).

3. Markets for consumption goods and labor clear.

4. The government budget constraint, (1.28), is satisfied.

The competitive equilibrium and the Pareto optimum are equivalent here, as in the version of the model without government. The social planner’s problem is

\[
\max_{c,\ell} u(c, \ell)
\]

subject to

\[ c + g = z(1 - \ell) \]

Substituting for \( c \) in the objective function, and maximizing with respect to \( \ell \), the first-order condition for this problem yields an equation which solves for \( \ell \):

\[
-uzu_1[z(1 - \ell) - g, \ell] + u_2[z(1 - \ell) - g, \ell] = 0. \tag{1.29}
\]

In Figure 1.4, the economy’s resource constraint is AB, and the Pareto optimum (competitive equilibrium) is D. Note that the slope of the resource constraint is \( -z = -w \).

We can now ask what the effect of a change in government expenditures would be on consumption and employment. In Figure 1.5, \( g \) increases from \( g_1 \) to \( g_2 \), shifting in the resource constraint. Given the government budget constraint, there is an increase in taxes, which represents a pure income effect for the consumer. Given that leisure and consumption are normal goods, quantities of both goods will decrease. Thus, there is crowding out of private consumption, but note that the decrease in consumption is smaller than the increase in government purchases, so that output increases. Algebraically, totally differentiate (1.29) and the equation \( c = z(1 - \ell) - g \) and solve to obtain

\[
\frac{d\ell}{dg} = \frac{-zu_{11} + u_{12}}{z^2u_{11} - 2zu_{12} + u_{22}} < 0
\]

\[
\frac{dc}{dg} = \frac{zu_{12} - u_{22}}{z^2u_{11} - 2zu_{12} + u_{22}} < 0 \tag{1.30}
\]
CHAPTER 1. SIMPLE REPRESENTATIVE AGENT MODELS

Here, the inequalities hold provided that $-zu_{11} + u_{12} > 0$ and $zu_{12} - u_{22} > 0$, i.e. if leisure and consumption are, respectively, normal goods. Note that (1.30) also implies that $\frac{dy}{dg} < 1$, i.e. the “balanced budget multiplier” is less than 1.

1.3 A “Dynamic” Economy

We will introduce some simple dynamics to our model in this section. The dynamics are restricted to the government’s financing decisions; there are really no dynamic elements in terms of real resource allocation, i.e. the social planner’s problem will break down into a series of static optimization problems. This model will be useful for studying the effects of changes in the timing of taxes.

Here, we deal with an infinite horizon economy, where the representative consumer maximizes time-separable utility,

$$\sum_{t=0}^{\infty} \beta^t u(c_t, \ell_t),$$
where $\beta$ is the discount factor, $0 < \beta < 1$. Letting $\delta$ denote the discount rate, we have $\beta = \frac{1}{1+\delta}$, where $\delta > 0$. Each period, the consumer is endowed with one unit of time. There is a representative firm which produces output according to the production function $y_t = z_t n_t$. The government purchases $g_t$ units of consumption goods in period $t$, $t = 0, 1, 2, \ldots$, and these purchases are destroyed. Government purchases are financed through lump-sum taxation and by issuing one-period government bonds. The government budget constraint is

$$g_t + (1 + r_t)b_t = \tau_t + b_{t+1}, \quad (1.31)$$

$t = 0, 1, 2, \ldots$, where $b_t$ is the number of one-period bonds issued by the government in period $t - 1$. A bond issued in period $t$ is a claim to $1 + r_{t+1}$ units of consumption in period $t+1$, where $r_{t+1}$ is the one-period interest rate. Equation (1.31) states that government purchases plus principal and interest on the government debt is equal to tax revenues plus new bond issues. Here, $b_0 = 0$. 
CHAPTER 1. SIMPLE REPRESENTATIVE AGENT MODELS

The optimization problem solved by the representative consumer is

\[
\max_{\{s_{t+1}, c_t, \ell_t\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t u(c_t, \ell_t)
\]

subject to

\[
c_t = w_t(1 - \ell_t) - \tau_t - s_{t+1} + (1 + r_t)s_t,
\]

for \(t = 0, 1, 2, \ldots, s_0 = 0\), where \(s_{t+1}\) is the quantity of bonds purchased by the consumer in period \(t\), which come due in period \(t+1\). Here, we permit the representative consumer to issue private bonds which are perfect substitutes for government bonds.

We will assume that

\[
\lim_{n \to \infty} \frac{s_n}{\prod_{i=1}^{n-1} (1 + r_i)} = 0,
\]

which states that the quantity of debt, discounted to \(t = 0\), must equal zero in the limit. This condition rules out infinite borrowing or “Ponzi schemes,” and implies that we can write the sequence of budget constraints, (1.32) as a single intertemporal budget constraint. Repeated substitution using (1.32) gives

\[
c_0 + \sum_{t=1}^\infty \frac{c_t}{\Pi_{i=1}^t (1 + r_i)} = w_0(1 - \ell_0) - \tau_0 + \sum_{t=1}^\infty \frac{w_t(1 - \ell_t) - \tau_t}{\Pi_{i=1}^t (1 + r_i)}.
\]

Now, maximizing utility subject to the above intertemporal budget constraint, we obtain the following first-order conditions.

\[
\beta^t u_1(c_t, \ell_t) - \frac{\lambda}{\Pi_{i=1}^t (1 + r_i)} = 0, \quad t = 1, 2, 3, 
\]

\[
\beta^t u_2(c_t, \ell_t) - \frac{\lambda w_t}{\Pi_{i=1}^t (1 + r_i)} = 0, \quad t = 1, 2, 3, 
\]

\[
u_1(c_0, \ell_0) - \lambda = 0
\]

\[
u_2(c_0, \ell_0) - \lambda w_0 = 0
\]

Here, \(\lambda\) is the Lagrange multiplier associated with the consumer’s intertemporal budget constraint. We then obtain

\[
\frac{u_2(c_t, \ell_t)}{u_1(c_t, \ell_t)} = w_t,
\]

(1.35)
1.3. A “DYNAMIC” ECONOMY

i.e. the marginal rate of substitution of leisure for consumption in any period equals the wage rate, and

\[
\frac{\beta u_1(c_{t+1}, \ell_{t+1})}{u_1(c_t, \ell_t)} = \frac{1}{1 + r_{t+1}},
\]

i.e. the intertemporal marginal rate of substitution of consumption equals the inverse of one plus the interest rate.

The representative firm simply maximizes profits in each period, i.e. it solves

\[
\max_{n_t} (z_t - w_t)n_t,
\]

and labor demand, \(n_t\), is perfectly elastic at \(w_t = z_t\).

A competitive equilibrium consists of quantities, \(\{c_t, \ell_t, n_t, s_{t+1}, b_{t+1}, \tau_t\}_{t=0}^\infty\), and prices \(\{w_t, r_{t+1}\}_{t=0}^\infty\) satisfying the following conditions.

1. Consumers choose \(\{c_t, \ell_t, s_{t+1}\}_{t=0}^\infty\) optimally given \(\{\tau_t\}\) and \(\{w_t, r_{t+1}\}_{t=0}^\infty\).
2. Firms choose \(\{n_t\}_{t=0}^\infty\) optimally given \(\{w_t\}_{t=0}^\infty\).
3. Given \(\{g_t\}_{t=0}^\infty\), \(\{b_{t+1}, \tau_t\}_{t=0}^\infty\) satisfies the sequence of government budget constraints (1.31).
4. Markets for consumption goods, labor, and bonds clear. Walras’ law permits us to drop the consumption goods market from consideration, giving us two market-clearing conditions:

\[
s_{t+1} = b_{t+1}, \quad t = 0, 1, 2, ..., \tag{1.37}
\]

and

\[
1 - \ell_t = n_t, \quad t = 0, 1, 2, ...
\]

Now, (1.33) and (1.37) imply that we can write the sequence of government budget constraints as a single intertemporal government budget constraint (through repeated substitution):

\[
g_0 + \sum_{t=1}^\infty \frac{g_t}{\prod_{i=1}^t (1 + r_i)} = \tau_0 + \sum_{t=1}^\infty \frac{\tau_t}{\prod_{i=1}^t (1 + r_i)}, \tag{1.38}
\]

i.e. the present discounted value of government purchases equals the present discounted value of tax revenues. Now, since the government
CHAPTER 1. SIMPLE REPRESENTATIVE AGENT MODELS

budget constraint must hold in equilibrium, we can use (1.38) to substitute in (1.34) to obtain

\[ c_0 + \sum_{t=1}^{\infty} \frac{c_t}{\Pi'} \frac{1}{1 + r_t} = w_0 (1 - \ell_0) - g_0 + \sum_{t=1}^{\infty} \frac{w_t (1 - \ell_t) - g_t}{\Pi'} \frac{1}{1 + r_t}. \]  

(1.39)

Now, suppose that \( \{w_t, r_{t+1}\}_{t=0}^\infty \) are competitive equilibrium prices. Then, (1.39) implies that the optimizing choices given those prices remain optimal given any sequence \( \{\tau_t\}_{t=0}^\infty \) satisfying (1.38). Also, the representative firm’s choices are invariant. That is, all that is relevant for the determination of consumption, leisure, and prices, is the present discounted value of government purchases, and the timing of taxes is irrelevant. This is a version of the Ricardian Equivalence Theorem. For example, holding the path of government purchases constant, if the representative consumer receives a tax cut today, he/she knows that the government will have to make this up with higher future taxes. The government issues more debt today to finance an increase in the government deficit, and private saving increases by an equal amount, since the representative consumer saves more to pay the higher taxes in the future.

Another way to show the Ricardian equivalence result here comes from computing the competitive equilibrium as the solution to a social planner’s problem, i.e.

\[ \max_{\{\ell_t\}_{t=0}^\infty} \sum_{t=0}^{\infty} \beta^t u[z_t (1 - \ell_t) - g_t, \ell_t] \]

This breaks down into a series of static problems, and the first-order conditions for an optimum are

\[ -z_t u_1[z_t (1 - \ell_t) - g_t, \ell_t] + u_2[z_t (1 - \ell_t) - g_t, \ell_t] = 0, \]

(1.40)

\( t = 0, 1, 2, \ldots \). Here, (1.40) solves for \( \ell_t, t = 0, 1, 2, \ldots \), and we can solve for \( c_t \) from \( c_t = z_t (1 - \ell_t) \). Then, (1.35) and (1.36) determine prices. Here, it is clear that the timing of taxes is irrelevant to determining the competitive equilibrium, though Ricardian equivalence holds in much more general settings where competitive equilibria are not Pareto optimal, and where the dynamics are more complicated.

Some assumptions which are critical to the Ricardian equivalence result are:
1.3. A “DYNAMIC” ECONOMY

1. Taxes are lump sum
2. Consumers are infinite-lived.
3. Capital markets are perfect, i.e. the interest rate at which private agents can borrow and lend is the same as the interest rate at which the government borrows and lends.
4. There are no distributional effects of taxation. That is, the present discounted value of each individual’s tax burden is unaffected by changes in the timing of aggregate taxation.