Economic Growth, Fall 2010
Solutions to Problem Set 1
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1. (**3 points**) Suppose that the growth rate of some variable, X, is constant and equal to $a > 0$ from time 0 to time t_1; drops to $0 < a_1 < a$ at time t_1; rises gradually from a_1 to a from time t_1 to time t_2; and is constant and equal to a after time t_2.

    (a) (**1 points**) Sketch a graph of the growth rate of X as a function of time.

    (b) (**2 points**) Sketch a graph of natural log of $X(t)$, $\log X(t)$, as a function of time.

2. (**5 points**) Consider a Solow model without technological progress, in its steady state. Now suppose that the rate of population growth falls.

    (a) (**3 points**) What happens to the steady-state values of capital per worker, output per worker, and consumption per worker? Sketch the paths of these variables as the economy moves to its new steady state.

    (b) (**2 points**) Sketch a graph of $\log Y(t)$ and $\log \frac{Y(t)}{L(t)}$ against time.

    Answer:

    (a) The new steady-state values of capital per worker, output per worker, and consumption per worker will be all higher.

3. (**10 points**) Assume the "Solow" economy is currently in the steady state. The savings rate $s = 20\%$; production function is $Y = K^{4/10}L^{6/10}$; population growth rate, n and depreciation rate, δ are some positive constants. (You don't need to know what they are to answer the questions, but you may pick any numbers if it helps.) The government decides to tax both wage and capital income at the proportional rate $0 < \tau < 1$. Thus, consumers receive real income in the amount equal to $(1 - \tau)Y$. Assume that the government invests the full amount of the tax proceedings. Thus, government savings are τY.

    (a) (**5 points**) Would you observe any changes in the economy resulting from the tax policy and, if yes, <u>what</u> changes (steady-state output per worker? steady-state capital per worker?)? Show a graph.

    (b) (**5 points**) If the governments wants to set τ to move the economy towards the golden-rule steady-state, what value of τ would it choose?

Answer:

(a) Households' total income in the economy is $(1-\tau)Y$; thus, the total private savings in the economy are $s(1-\tau)Y$. Assuming that government spends all tax proceedings on purchasing investment goods, public savings are τY. Total savings in the economy are $s(1-\tau)Y + \tau Y = sY + \tau \underbrace{(1-s)}_{>0} Y > sY$—total savings and investment in the economy are larger than before the introduction of taxes. Thus, the investment curve, after the introduction of proportional taxes, lies above the investment curve before taxes; since the break-even investment lines are unaffected the economy will reach the new steady-state with higher capital-per worker ratio and high output per worker.

(b) The golden-rule steady state is achieved when the economy's effective savings rate equals 0.4. The effective savings rate equals $s + \tau(1-s)$. Thus, the tax rate should be

$$\tau = \frac{0.4 - s}{1 - s} = \frac{0.4 - 0.2}{1 - 0.2} = \frac{1}{4} = 25\%.$$

4. (**6 points**) Consider the Solow growth model with constant savings rate s and depreciation rate of capital equal to δ. Assume that population is constant and the aggregate production function is given by the constant returns to scale production function

$$F\left(A_K(t)K(t), A_L(t)L(t)\right),$$

where $\dot{A}_L(t)/A_L(t) = g_L > 0$ and $\dot{A}_K(t)/A_K(t) = g_K > 0$. Suppose that F is Cobb-Douglas. Determine the balanced-path growth rate in output per worker, the steady-state value of capital per effective worker, and the law of motion of capital per effective worker. (Assume that production function is of the form $Y = (A_K K)^\alpha (A_L L)^{1-\alpha}$. The trick is to define the labor-augmenting technological progress by choosing $A(t)$ which will be a mix of $A_K(t)$ and $A_L(t)$ as we've done in class, and then proceed as usual.)

Answer:

We can redefine our production function so that technological progress is of labor-augmenting type: $Y = K^\alpha (AL)^{1-\alpha}$. We have to set $A_K^\alpha A_L^{1-\alpha} = A^{1-\alpha}$, or

$$A = A_K^{\alpha/(1-\alpha)} A_L.$$

The rest of the analysis is usual; define $k = \frac{K}{AL}$ and $y = \frac{Y}{AL} = k^\alpha$. The law of motion of capital per effective worker will be obtained from

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L}.$$

Note that $\frac{\dot{A}}{A} = \frac{\alpha}{1-\alpha}\frac{\dot{A}_K}{A_K} + \frac{\dot{A}_L}{A_L} = \frac{\alpha}{1-\alpha}g_K + g_L$. Utilizing our assumption on fixed labor force,

$$\begin{aligned}
\frac{\dot{k}}{k} &= \frac{\dot{K}}{K} - \left(\frac{\alpha}{1-\alpha}g_K + g_L\right) = \frac{sY - \delta K}{K} - \left(\frac{\alpha}{1-\alpha}g_K + g_L\right) \\
&= \frac{sY/(AL)}{K/(AL)} - \left(\delta + \frac{\alpha}{1-\alpha}g_K + g_L\right) = \frac{sk^\alpha}{k} - \left(\delta + \frac{\alpha}{1-\alpha}g_K + g_L\right); \\
\dot{k} &= sk^\alpha - \left(\delta + \frac{\alpha}{1-\alpha}g_K + g_L\right)k.
\end{aligned}$$

Setting the law of motion of capital-labor ratio to zero

$$k^* = \left(\frac{s}{\delta + \frac{\alpha}{1-\alpha}g_K + g_L}\right)^{\frac{1}{1-\alpha}}.$$

Notice that $y = \frac{Y}{AL} = \frac{Y}{A_K^{\alpha/(1-\alpha)}A_L}$ does not grow in the steady state. Output per worker can be expressed as

$$\frac{Y}{L} = yA = yA_K^{\alpha/(1-\alpha)}A_L.$$

Since y does not grow, output per worker grows at the rate $\frac{\alpha}{1-\alpha}g_K + g_L$ on a balanced growth path.