Chapter 2. The Solow Growth Model

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Solow model

Solow model is a starting point for more complex models.

Abstracts from modeling heterogeneous households (in tastes, abilities, etc.), heterogeneous sectors in the economy, and social interactions. It is a one-good economy with simplified individual decisions.

We'll discuss the Solow model both in discrete and continuous time.

Households and production

- Closed economy, with a unique final good.
- Time is discrete: $t = 0, 1, 2, \ldots, \infty$ (days/weeks/months/years).
- A representative household saves a constant fraction $s \in (0, 1)$ of its disposable income (a bit unrealistic).
- A representative firm with the aggregate production function $Y(t) = F(K(t), L(t), A(t))$.

Production function

$$Y(t) = F(K(t), L(t), A(t)),$$

where $Y(t)$ is the total production of the final good at time t, $K(t)$—the capital stock at time t, and $A(t)$ is technology at time t.

Think of L as hours of employment or the number of employees; K the quantity of "machines" (K is the same as the final good in the model; e.g., K is "corn"); A is a shifter of the production function (a broad notion of technology: the effects of the organization of production and of markets on the efficiency of utilization of K and L).

Assumption on technology

Technology is assumed to be *free*. That is, it is publicly available and

- *Non-rival*—consumption or use by others does not preclude an individual's consumption or use.
- *Non-excludable*—impossible to prevent another person from using or consuming it.

E.g., society's knowledge of how to use the wheels.

Assumptions on production function

The production function $F(K, L, A)$ is twice differentiable in K and L, satisfying:

$$F_K(K, L, A) = \frac{\partial F(K, L, A)}{\partial K} > 0, \; F_L(K, L, A) = \frac{\partial F(K, L, A)}{\partial L} > 0$$

$$F_{KK} = \frac{\partial^2 F(K, L, A)}{\partial K^2} < 0, \; F_{LL} = \frac{\partial^2 F(K, L, A)}{\partial L^2} < 0.$$

F is constant returns to scale in K and L.

$F_K > 0, F_L > 0$: positive marginal products (the level of production increases with the amount of inputs used);
$F_{KK} < 0, F_{LL} < 0$: diminishing marginal products (more labor, holding other things constant, increases output by less and less).

Constant returns to scale assumption

$F(K, L, A)$ is constant returns to scale in K and L if $\lambda F(K, L, A) = F(\lambda K, \lambda L, A)$. In this case, $F(K, L, A)$ is also said to be *homogeneous* of degree one.

In general, a function $F(K, L, A)$ is homogeneous of degree m in K and L if $\lambda^m F(K, L, A) = F(\lambda K, \lambda L, A)$.

Euler's theorem

Suppose that $g(x, y, z)$ is differentiable in x and y and homogeneous of degree m in x and y, with partial first derivatives g_x and g_y. Then, (1)

$$mg(x, y, z) = g_x(x, y, z)x + g_y(x, y, z)y,$$

and (2) g_x and g_y are homogeneous of degree $m - 1$ in x and y.

Proof [of part (1)]. By definition $\lambda^m g(x, y, z) = g(\lambda x, \lambda y, z)$. Differentiate both sides wrt λ to obtain
$m\lambda^{m-1}g(x, y, z) = g_x(\lambda x, \lambda y, z)x + g_y(\lambda x, \lambda y, z)y$, for any λ. Set $\lambda \equiv 1$.
Thus, $mg(x, y, z) = g_x(x, y, z)x + g_y(x, y, z)y$.

Endowments, market structure and market clearing

1. Competitive markets.
2. Households own labor, supplied inelastically: all the endowment of labor $\overline{L}(t)$ is supplied at a *non-negative* rental price.
    - Market clearing condition: $L^d(t) = \overline{L}(t)$. Rental price of labor at t is the wage rate at t, $w(t)$.
3. Households own the capital stock and rent it to firms. $R(t)$ is the rental price of capital at t.
    - Market clearing condition: $K^d(t) = K^s(t)$.
    - K_t^d used in production at t is consistent with households' endowments and saving behavior.
4. $K(0) \geq 0$—initial capital stock is given.
5. The price of the final good, P_t, is normalized to 1 at all t.
6. Capital depreciates at the rate $\delta \in (0,1)$ (machines utilized in production lose some of their value due to wear and tear). 1 unit now "becomes" $1 - \delta$ units next period.
7. $R(t) + (1 - \delta) = 1 + r(t)$, where $r(t)$ is the real interest rate at t.

Firm optimization and equilibrium

Firms' objective, at each t, is to maximize profits; firms are competitive in product and rental markets, i.e., take $w(t)$ and $R(t)$ as given.

$$\max_{K\geq 0, L\geq 0} \pi = F(K^d, L^d, A(t)) - R(t)K^d - w(t)L^d,$$

where superscript d stands for demand, and π is profit.

Note that maximization is in terms of aggregate variables (due to the representative firm assumption), and $R(t)$ and $w(t)$ are relative prices of labor and capital in terms of the final good.

Since F is constant returns to scale, there is no well-defined solution (K^{*d}, L^{*d}) (* means optimal, or profit-maximizing).
If $(K^{*d} \geq 0, L^{*d} \geq 0)$ and $\pi > 0$, then $(\lambda K^{*d} \geq 0, \lambda L^{*d} \geq 0)$ will bring $\lambda\pi$.
Thus, want to raise K and L as much as possible. A trivial solution $K = L = 0$, or multiple values of K and L that give $\pi = 0$. When profits are zero, would rent L and K so that $L = L^s(t)$ and $K = K^s(t)$.

Factor prices

$$w(t) = F_L(K(t), L(t), A(t)),$$
$$R(t) = F_K(K(t), L(t), A(t)).$$

Using these two equilibrium conditions and Euler's theorem, we can prove that firms make zero profits in the Solow growth model. $\underbrace{Y(t)}_{LHS} = F(K(t), L(t), A(t)) = F_K(K(t), L(t), A(t)) \times K(t) + F_L(K(t), L(t), A(t)) \times L(t) = \underbrace{R(t)K(t) + w(t)L(t)}_{RHS}.$

Factor payments (RHS) exhaust total revenue (LHS), hence profits are zero.

Inada conditions

We'll assume that F satisfies *Inada conditions*:

$$\lim_{L\to 0} F_L(K,L,A) = \infty,$$

$$\lim_{K\to 0} F_K(K,L,A) = \infty,$$

$$\lim_{L\to \infty} F_L(K,L,A) = 0,$$

$$\lim_{K\to \infty} F_K(K,L,A) = 0.$$

The first two conditions say that the "first" units of labor and capital are highly productive, the last two—that when labor and capital are sufficiently abundant, the use of a marginal unit brings zero output.

The law of motion of the capital stock

The law of motion of $K(t)$ is given by

$$K(t+1) = I(t) + (1-\delta)K(t),$$

where $I(t)$ is investment at time t.

Recall the national income accounts identity for a closed economy:

$$Y(t) = C(t) + I(t).$$

Also, $S(t) = I(t) = Y(t) - C(t)$. By assumption, $S(t) = sY(t)$, and so $C(t) = Y(t) - sY(t) = (1-s)Y(t)$. Supply of capital resulting from households' behavior is $K^s(t+1) = (1-\delta)K(t) + S(t) = (1-\delta)K(t) + sY(t)$. Equilibrium conditions are: $K^s(t+1) = K(t+1)$ and $\bar{L}(t) = L(t)$. *The fundamental law of motion* of the Solow model is:

$$K(t+1) = (1-\delta)K(t) + sY(t) = (1-\delta)K(t) + sF(K(t), L(t), A(t)).$$

Definition of Equilibrium

Equilibrium is defined as an entire path of *allocations*, $C(t)$, $K(t)$, $Y(t)$, and *prices*, $w(t)$ and $R(t)$, given an initial level of capital stock $K(0)$, and given sequences $\{L(0), L(1), L(2), \ldots, L(t), \ldots\}$ and $\{A(0), A(1), A(2), \ldots, A(t), \ldots\}$.

Equilibrium without population growth and technological progress

In this case, $\overline{L}(t) = L$, and $A(t) = A$.
Example. The Cobb-Douglas production function.

$$Y(t) = F(K(t), L(t), A(t)) = AK(t)^{\alpha}L^{1-\alpha}, \;\; 0 < \alpha < 1.$$

Note that $F(\lambda K, \lambda L, A) = A(\lambda K(t))^{\alpha}(\lambda L)^{1-\alpha} = A\lambda^{\alpha}\lambda^{1-\alpha}K(t)^{\alpha}L^{1-\alpha} = \lambda AK(t)^{\alpha}L^{1-\alpha} = \lambda Y(t)$. Let $\lambda \equiv \frac{1}{L}$. Then
$\frac{Y(t)}{L} = A(\frac{K(t)}{L})^{\alpha}(\frac{L}{L})^{1-\alpha} = A(\frac{K(t)}{L})^{\alpha}1^{1-\alpha} = A(\frac{K(t)}{L})^{\alpha}$.
Define $k(t) \equiv \frac{K(t)}{L}$ and $y(t) \equiv \frac{Y(t)}{L}$. Then $y(t) = Ak(t)^{\alpha}$.
$R(t) = F_K = A\alpha K(t)^{\alpha-1}L^{1-\alpha} = A\alpha(\frac{K(t)}{L})^{\alpha-1} = A\alpha k(t)^{\alpha-1}$.
$w(t) = F_L = A(1-\alpha)K(t)^{\alpha}L^{-\alpha} = A(1-\alpha)(\frac{K(t)}{L})^{\alpha} = A(1-\alpha)k(t)^{\alpha}$. Note also that $w(t)L = Y(t) - R(t)K(t)$, and so $w(t) = \frac{Y(t)}{L} - R(t)k(t) = Ak(t)^{\alpha} - A\alpha k(t)^{\alpha-1}k(t) = Ak(t)^{\alpha} - A\alpha k(t)^{\alpha} = A(1-\alpha)k(t)^{\alpha} = (1-\alpha)y(t)$.

Steady-state equilibrium

Recall the law of motion of the capital stock:

$$K(t+1) = (1-\delta)K(t) + sF(K(t), L, A),$$

where we utilized our assumptions that technology and population are not growing. Divide the equation by L, to obtain,

$$k(t+1) = (1-\delta)k(t) + s\frac{Y(t)}{L} = (1-\delta)k(t) + sF(\frac{K(t)}{L}, 1, A).$$

Normalize A to one, and define $F(\frac{K(t)}{L}, 1, 1) = f(k(t))$. Thus, the law of motion in per worker terms is:

$$k(t+1) = (1-\delta)k(t) + sf(k(t)).$$

A steady-state equilibrium without population and technological growth is an equilibrium path so that $k(t) = k^*$ for all t.

Steady-state equilibrium—contd.

For a steady state,

$$k^* = sf(k^*) + (1-\delta)k^*$$
$$sf(k^*) = \delta k^*.$$

Also,

$$y^* = f(k^*)$$
$$c^* = (1-s)f(k^*).$$

Notes: since $k^* = \frac{K^*}{L}$ is constant and L is not growing, K is not growing in the steady state either; to keep capital per worker, k, constant investment per worker should be equal to the amount of capital per worker that needs to be "replenished" because of depreciation; k^* will be a function of s and δ, i.e., $k^* = k^*(s, \delta)$.

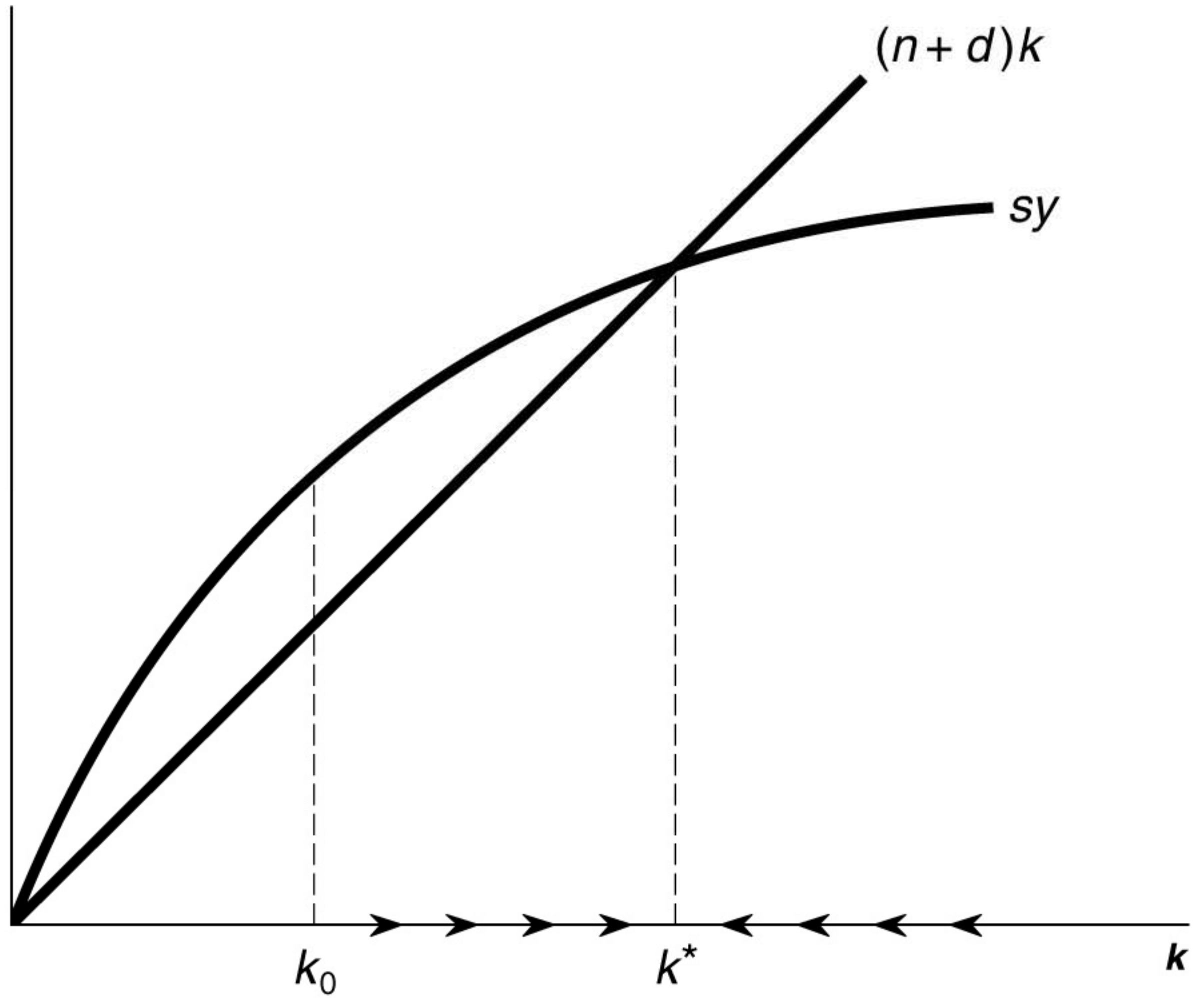

FIGURE 2.2 THE BASIC SOLOW DIAGRAM

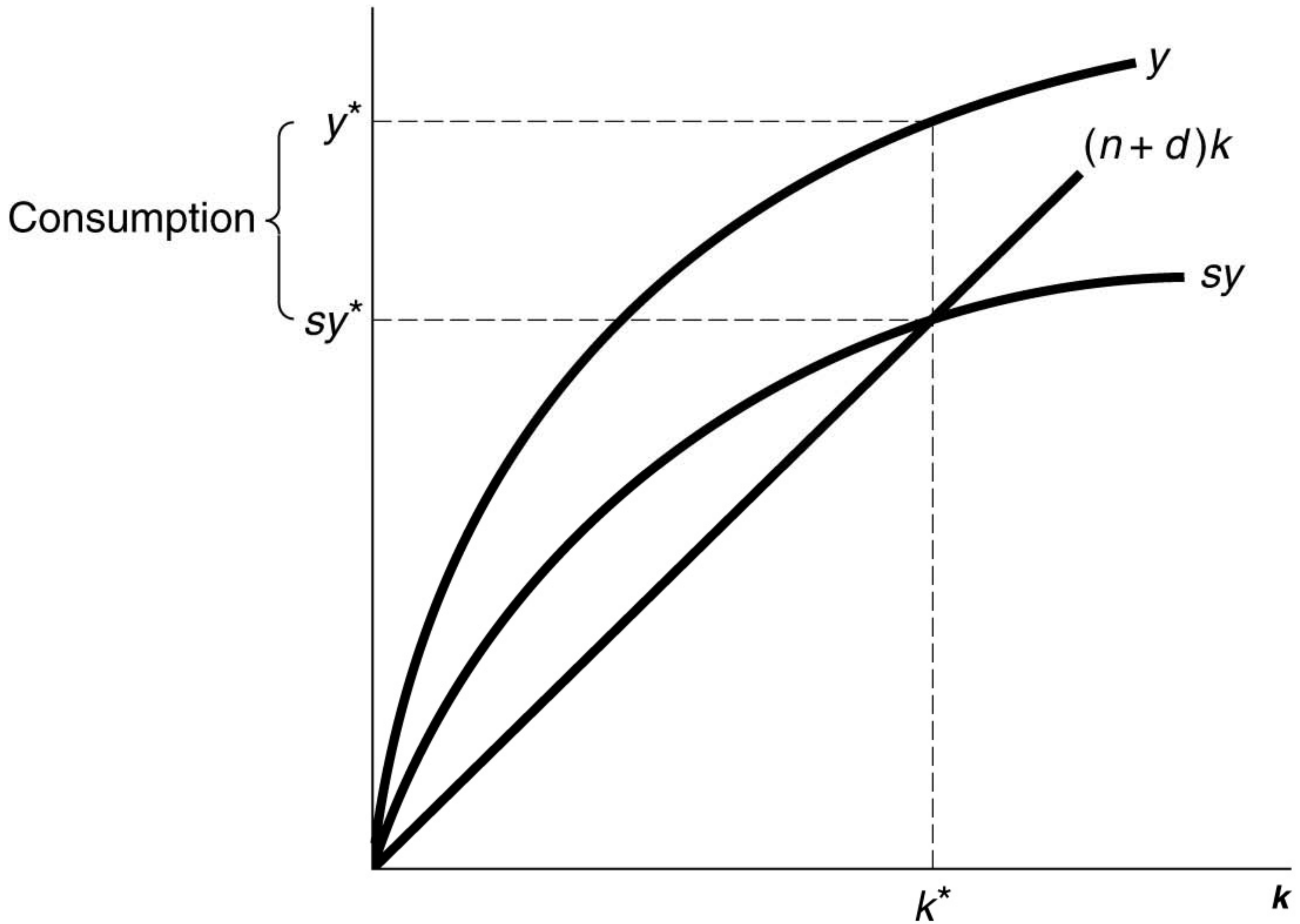

FIGURE 2.3 THE SOLOW DIAGRAM AND THE PRODUCTION FUNCTION

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Steady-state predictions

We normalized A to 1 before. Redefine our per worker production function as $\tilde{f}(k) = Af(k)$. The steady-state equilibrium will now be obtained from $sAf(k^*) = \delta f(k^*)$, and $k^* = k^*(s, A, \delta)$. Thus, $sAf(k^*(s, A, \delta)) = \delta f(k^*(s, A, \delta))$. It can be shown that

$$\frac{\partial k^*(s, A, \delta)}{\partial A} > 0, \quad \frac{\partial k^*(s, A, \delta)}{\partial s} > 0, \quad \frac{\partial k^*(s, A, \delta)}{\partial \delta} < 0$$
$$\frac{\partial y^*(s, A, \delta)}{\partial A} > 0, \quad \frac{\partial y^*(s, A, \delta)}{\partial s} > 0, \quad \frac{\partial y^*(s, A, \delta)}{\partial \delta} < 0.$$

Intuition: economies with higher saving rates and better technologies will accumulate higher capitals per worker and will be richer; higher depreciation rates lead to lower standards of living and lower capitals per worker.

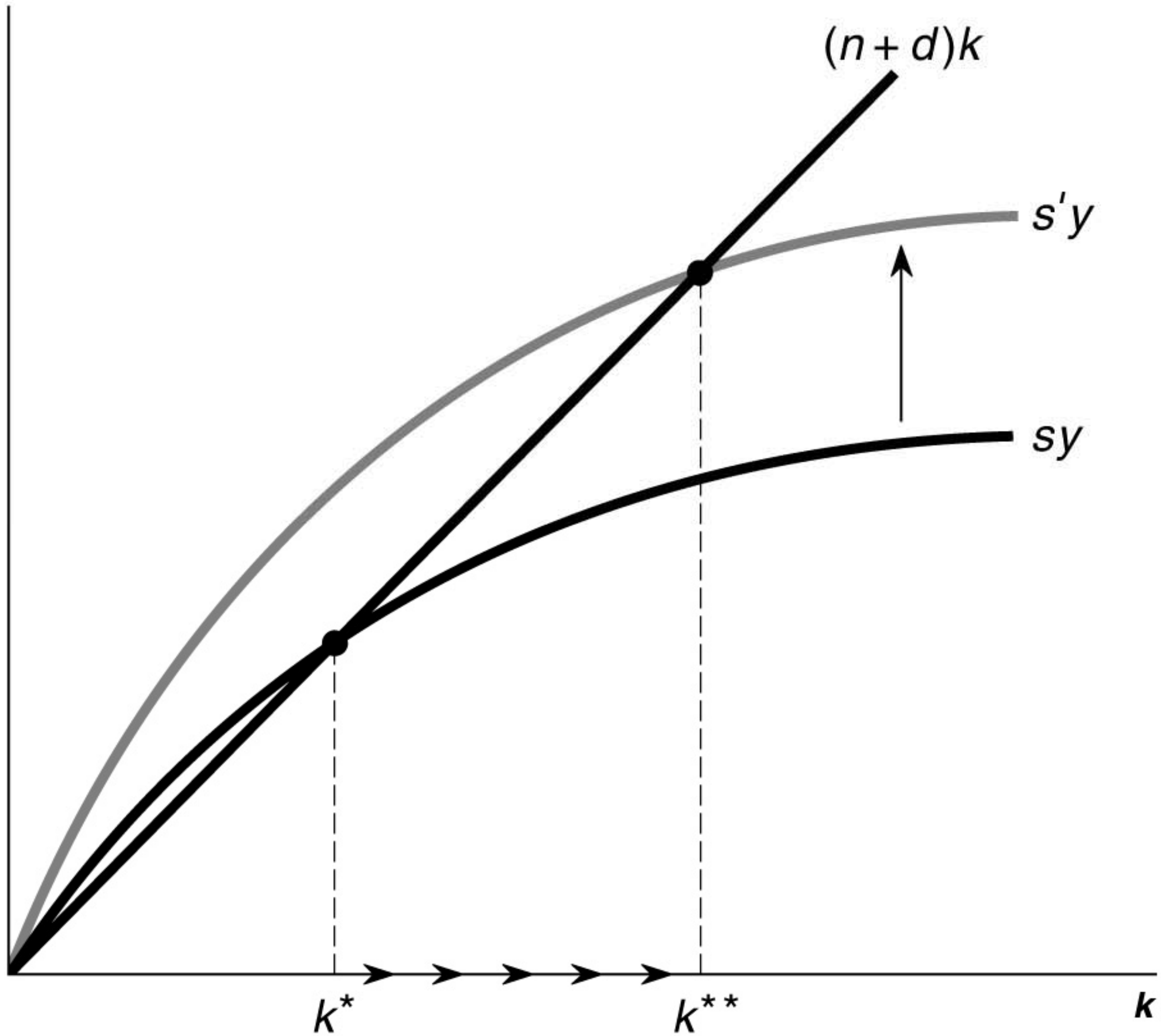

FIGURE 2.4 AN INCREASE IN THE INVESTMENT RATE

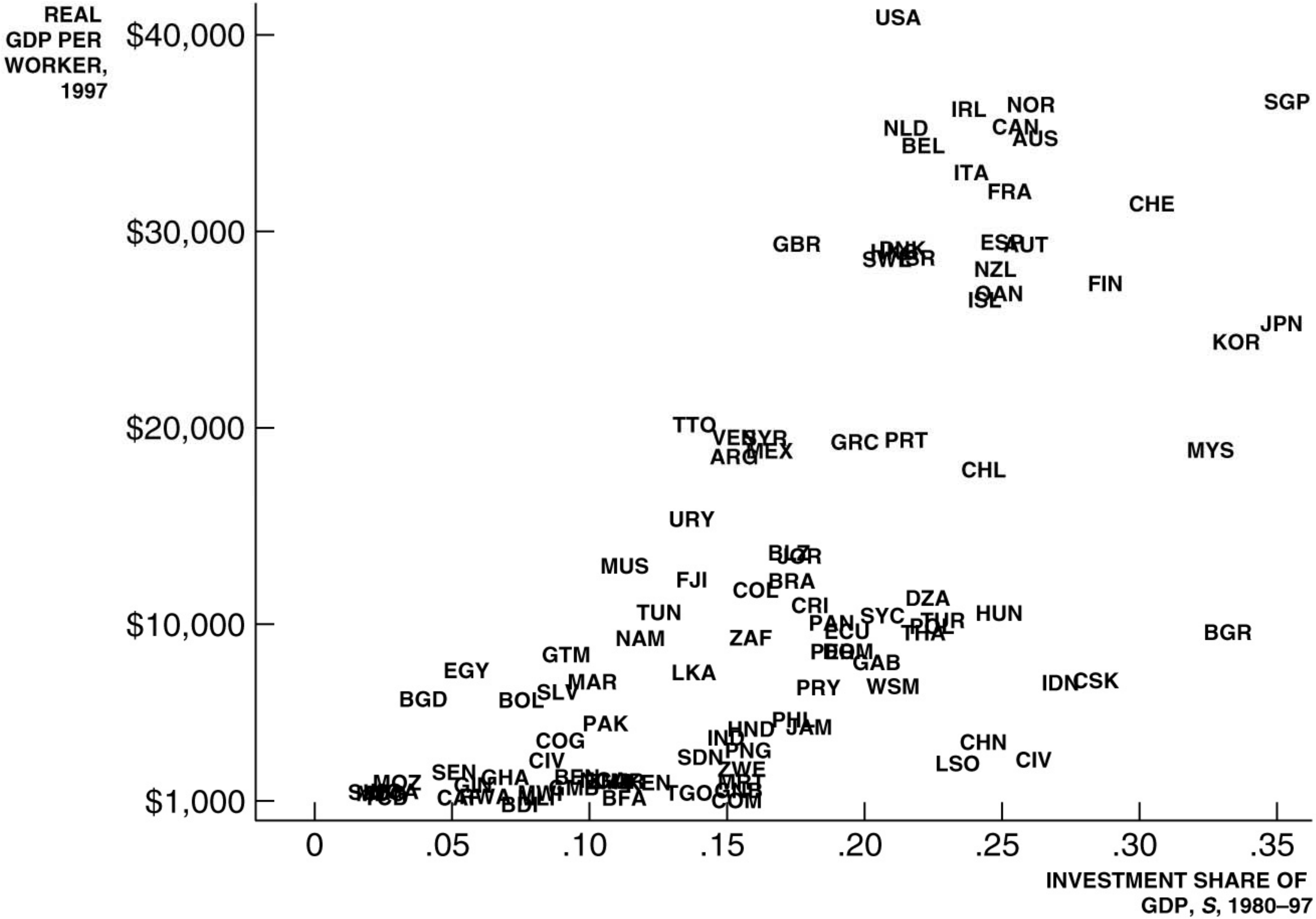

FIGURE 2.6 GDP PER WORKER VERSUS THE INVESTMENT RATE

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The Solow model in continuous time

Consider the following (difference) equation $x(t+1) - x(t) = g(x(t))$, where $g(\cdot)$ is some function. In terms of our law of motion, e.g., $x(t) = k(t)$, and $g(x(t)) \equiv sf(k(t)) - \delta k(t)$. This equation specifies the change in x between two discrete time points. What happens inside of the interval $[t, t+1]$ is "not known." Assume that we can partition time very finely. If t and $t+1$ are "close enough," the following approximation is reasonable

$$x(t+\Delta t) - x(t) \approx \Delta t g(x(t)).$$

Dividing both sides by Δt and taking the limits gives

$$\lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \dot{x}(t) \approx g(x(t)),$$

where $\dot{x}(t) = \frac{dx(t)}{dt}$—the time derivative of $x(t)$.

The law of motion of capital in continuous time

Now we refer to $w(t)$ and $R(t)$ as *instantaneous* rental rates (the flows received at instant t). Assume that the labor force grows at a constant rate n, $L(t) = \exp(nt)L(0)$. Note that $\log L(t) = nt + \log(L(0))$. If $z(t) = f(x(t))$, then $\dot{z}(t) = f_x \dot{x}(t)$ (the chain rule of differentiation). Since $\frac{\partial}{\partial x}(\log(x)) = \frac{1}{x}$, then defining $z(t) \equiv \log(L(t))$ and differentiating $z(t)$ with respect to time gives $\dot{z}(t) = \frac{1}{L(t)}\dot{L}(t) = \frac{\dot{L}(t)}{L(t)}$. Differentiating $nt + L(0)$ with respect to time gives n. Thus, $\frac{\dot{L}(t)}{L(t)} = n$.

Define $k(t) = \frac{K(t)}{L(t)}$. Taking natural logs from both sides gives $\log(k(t)) = \log(K(t)) - \log(L(t))$. Taking time derivatives from both sides gives $\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)} = \frac{\dot{K}(t)}{K(t)} - n = \frac{sF(K(t),L(t),A) - \delta K(t)}{K(t)} - n = \frac{sF(K(t),L(t),A)}{K(t)} - (n+\delta) = \frac{sF(K(t),L(t),A)/L(t)}{K(t)/L(t)} - (n+\delta) = \frac{sf(k(t))}{k(t)} - (n+\delta)$. Thus,

$$\dot{k}(t) = sf(k(t)) - (n+\delta)k(t).$$

Again, the steady-state occurs when $\dot{k}(t) = 0$, and $sf(k^*) = (n+\delta)k^*$.

Add-on to the steady-state predictions

Since we introduced population growth into the model, $k^* = k^*(s, A, \delta, n)$, and $y^* = y^*(s, A, \delta, n)$. We can show that

$$\frac{\partial k^*(s, A, \delta, n)}{\partial n} < 0, \quad \frac{\partial y^*(s, A, \delta, n)}{\partial n} < 0.$$

Economies with higher population growth rates have lower capital-per-worker ratios and lower incomes per worker.

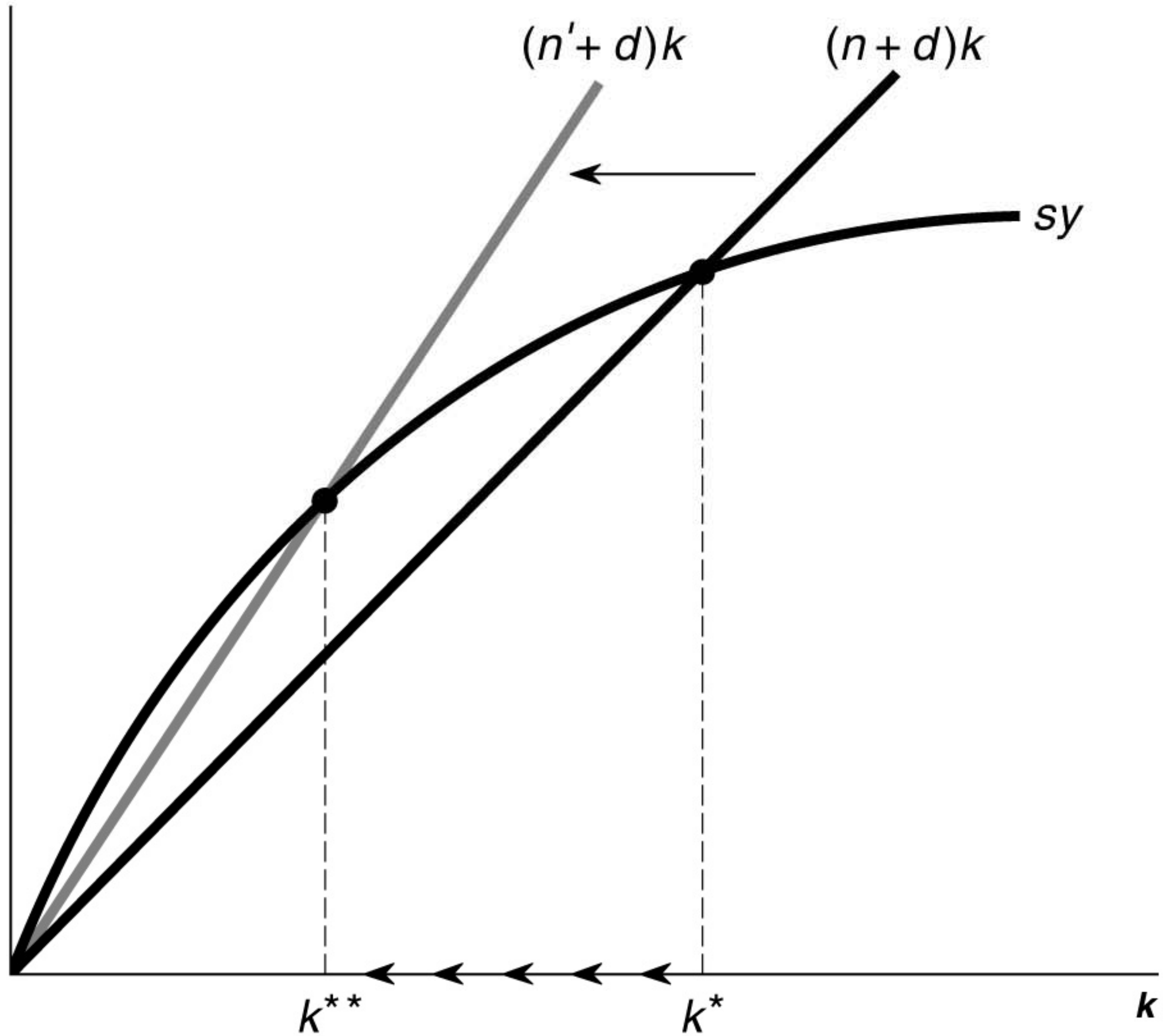

FIGURE 2.5 AN INCREASE IN POPULATION GROWTH

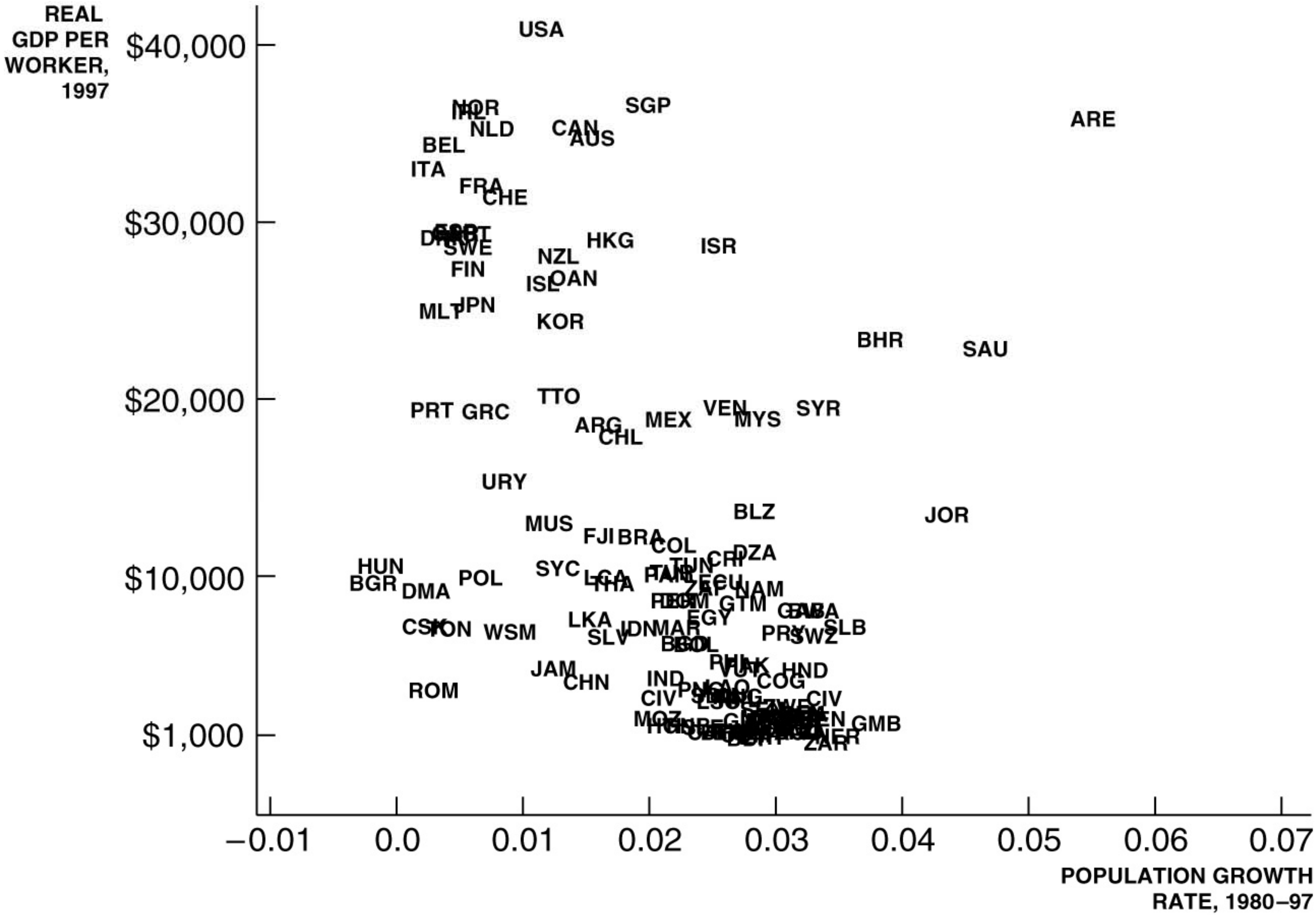

FIGURE 2.7 GDP PER WORKER VERSUS POPULATION GROWTH RATES

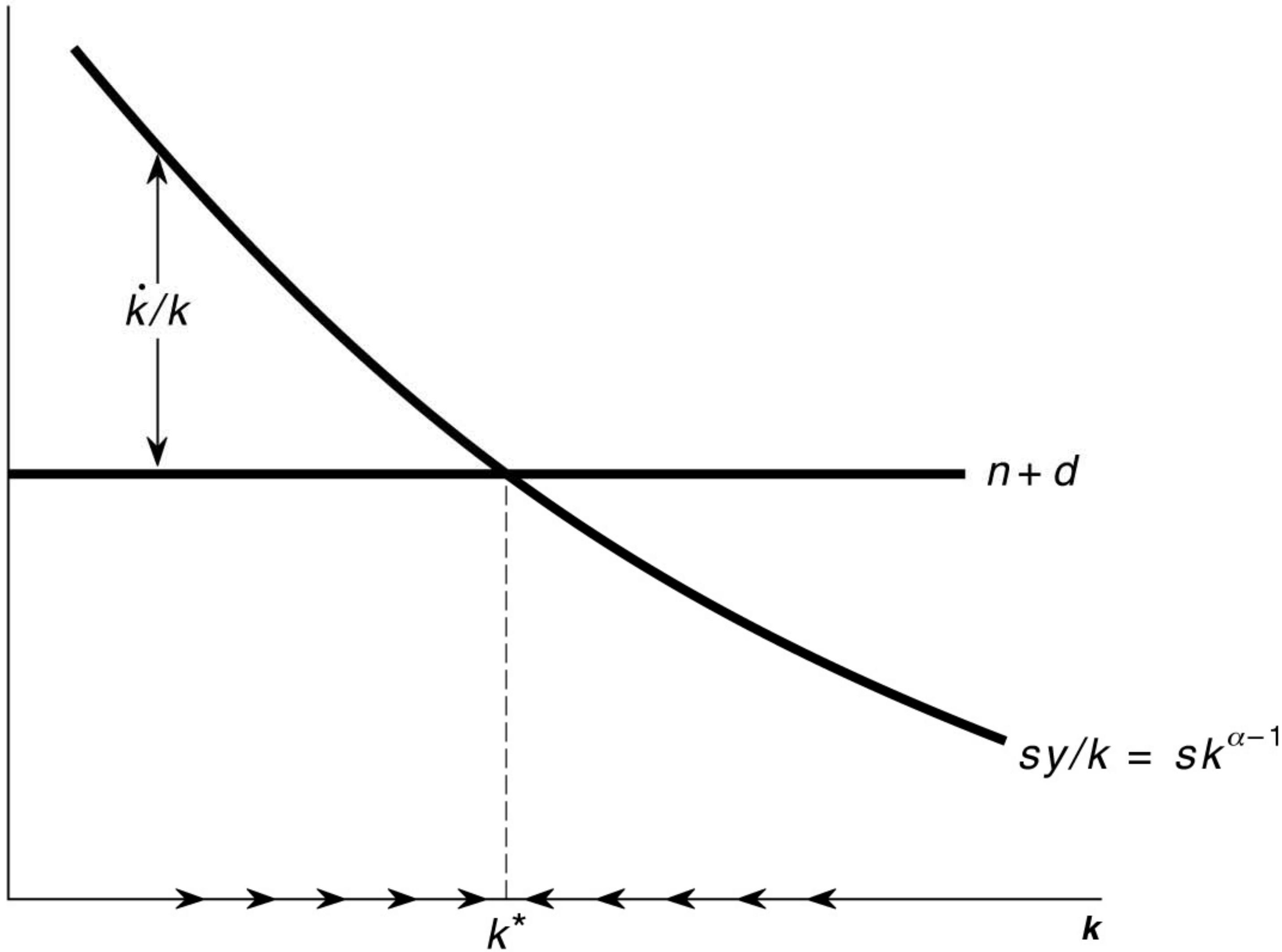

FIGURE 2.8 TRANSITION DYNAMICS

Factor shares in total income

Consider again the Cobb-Douglas production function $Y = AK^{\alpha}L^{1-\alpha}$, $0 < \alpha < 1$.

The share of capital costs in total income is defined as $\frac{R(t)K(t)}{Y(t)}$.

In competitive factor markets $R(t) = F_K(K(t), L(t), A(t))$ and $w(t) = F_L(K(t), L(t), A(t))$. Thus,

$\frac{R(t)K(t)}{Y(t)} = \frac{A\alpha K(t)^{\alpha-1}L(t)^{1-\alpha}K(t)}{Y(t)} = \frac{\alpha AK(t)^{\alpha}L(t)^{1-\alpha}}{Y(t)} = \frac{\alpha Y(t)}{Y(t)} = \alpha.$

From Euler theorem, $Y(t) = w(t)L(t) + R(t)K(t)$. Thus, $\frac{w(t)L(t)}{Y(t)} = 1 - \alpha$. For the Cobb-Douglas function, the shares of capital and labor costs in total income are constant.

For this function,

$y(t) = \frac{Y(t)}{L(t)} = \frac{AK(t)^{\alpha}L(t)^{1-\alpha}}{L(t)} = A(\frac{K(t)}{L(t)})^{\alpha} = Ak(t)^{\alpha}$. Thus,

$\dot{k}(t) = sAk(t)^{\alpha} - (n+\delta)k(t)$, and k^* occurs when $sA(k^*)^{\alpha} = (n+\delta)k^*$, or when $(k^*)^{1-\alpha} = \frac{sA}{n+\delta}$. Hence,

$k^* = \left(\frac{sA}{n+\delta}\right)^{\frac{1}{1-\alpha}}$.

Solow model with technological progress

Introduce sustained growth by allowing technological progress in the form of changes in $A(t)$.

How to model technological growth and its impact on $Y(t)$?

Introduce technological progress so that the resulting allocations ($Y(t)$, $K(t)$, and $C(t)$) are consistent with *balanced growth*, as defined by the Kaldor facts.

Types of technological progress

Define different types of “neutral” technological progress.

Hicks-neutral:

$$F(K(t), L(t), A(t)) = A(t)\tilde{F}(K(t), L(t)).$$

Solow-neutral (capital-augmenting):

$$F(K(t), L(t), A(t)) = \tilde{F}(A(t)K(t), L(t)).$$

Harrod-neutral (labor-augmenting):

$$F(K(t), L(t), A(t)) = \tilde{F}(K(t), A(t)L(t)).$$

Balanced growth is possible in the long-run only if technological progress is Harrod-neutral or labor-augmenting.

The Solow model with technological progress: continuous time

Let $Y(t) = F(K(t), A(t)L(t))$ be constant returns to scale in $K(t)$ and $L(t)$, and $\frac{\dot{A}(t)}{A(t)} = g > 0$. Define the aggregates as ratios to *effective labor*, $A(t)L(t)$. Thus, $k(t) = \frac{K(t)}{A(t)L(t)}$ is capital per effective labor, and $y(t) = \frac{Y(t)}{A(t)L(t)}$ is output per effective labor.

Note that $\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - g - n$;

$y(t) = \frac{Y(t)}{A(t)L(t)} = \frac{F(K(t),A(t)L(t))}{A(t)L(t)} = F(\frac{K(t)}{A(t)L(t)}, 1) = f(k(t))$, and

$\frac{Y(t)}{L(t)} = y(t)A(t) = A(t)f(k(t))$.

$$\begin{aligned}
\frac{\dot{k}(t)}{k(t)} &= \frac{sF(K(t), A(t)L(t)) - \delta K(t)}{K(t)} - (n+g) \\
&= \frac{sF(K(t), A(t)L(t))/(A(t)L(t))}{K(t)/(A(t)L(t))} - (\delta + n + g) \\
&= \frac{sf(k(t))}{k(t)} - (n + g + \delta).
\end{aligned}$$

An equilibrium is defined as before, now in terms of the constancy of $k(t) = \frac{K(t)}{A(t)L(t)}$. The growth rate of $\frac{K(t)}{L(t)}$, however, will be equal to g.

Equilibrium with technological progress

Consider the Solow growth model with labor-augmenting technological progress at the rate g and population growth at the rate n. Then there exists a unique balanced growth path where the effective capital-labor ratio is constant and given by $s(f(k^*)) = (n + g + \delta)k^*$, where $k = \frac{K}{AL}$. Output per worker, capital per worker and consumption per worker will grow at the rate g.

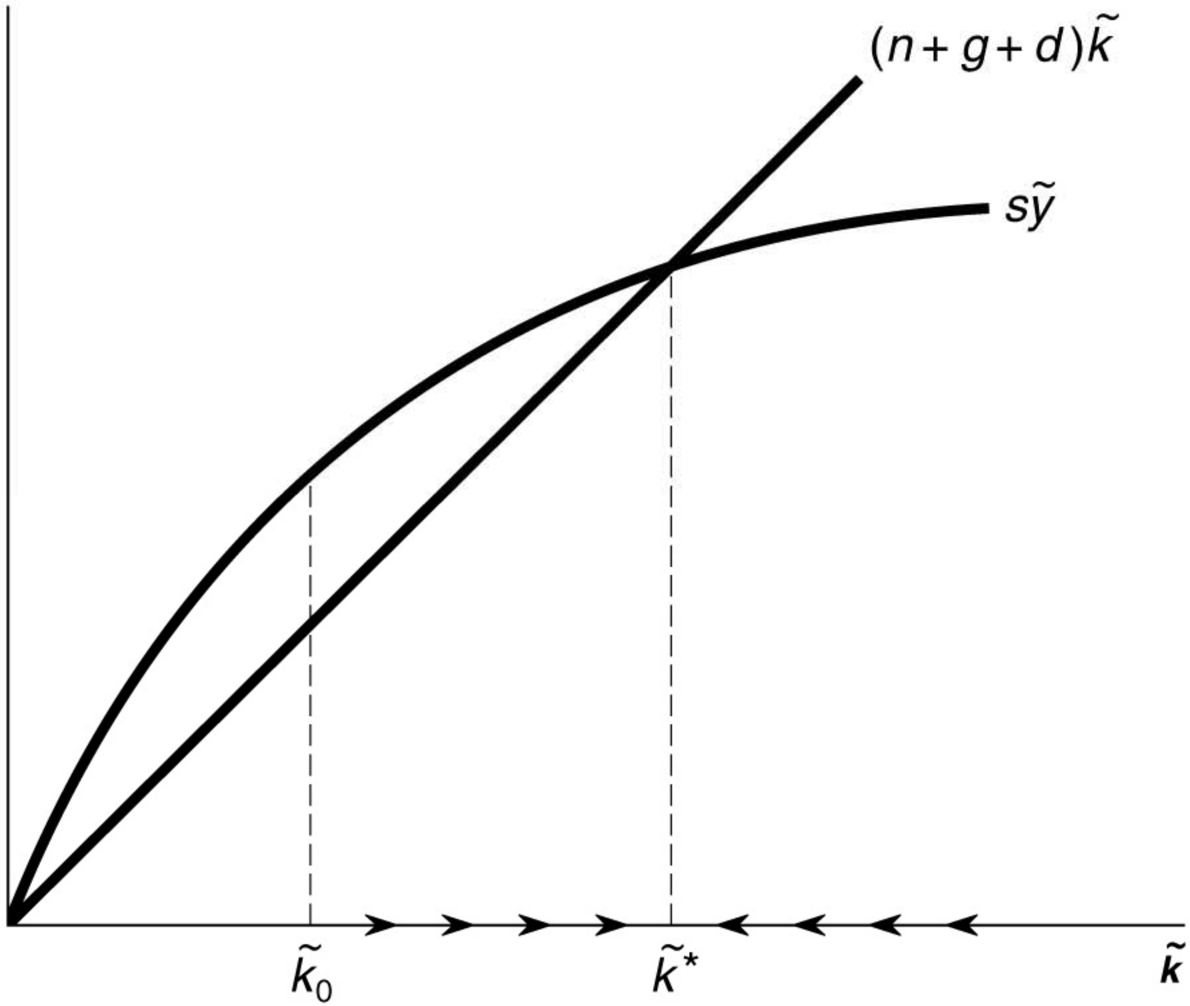

FIGURE 2.9 THE SOLOW DIAGRAM WITH TECHNOLOGICAL PROGRESS

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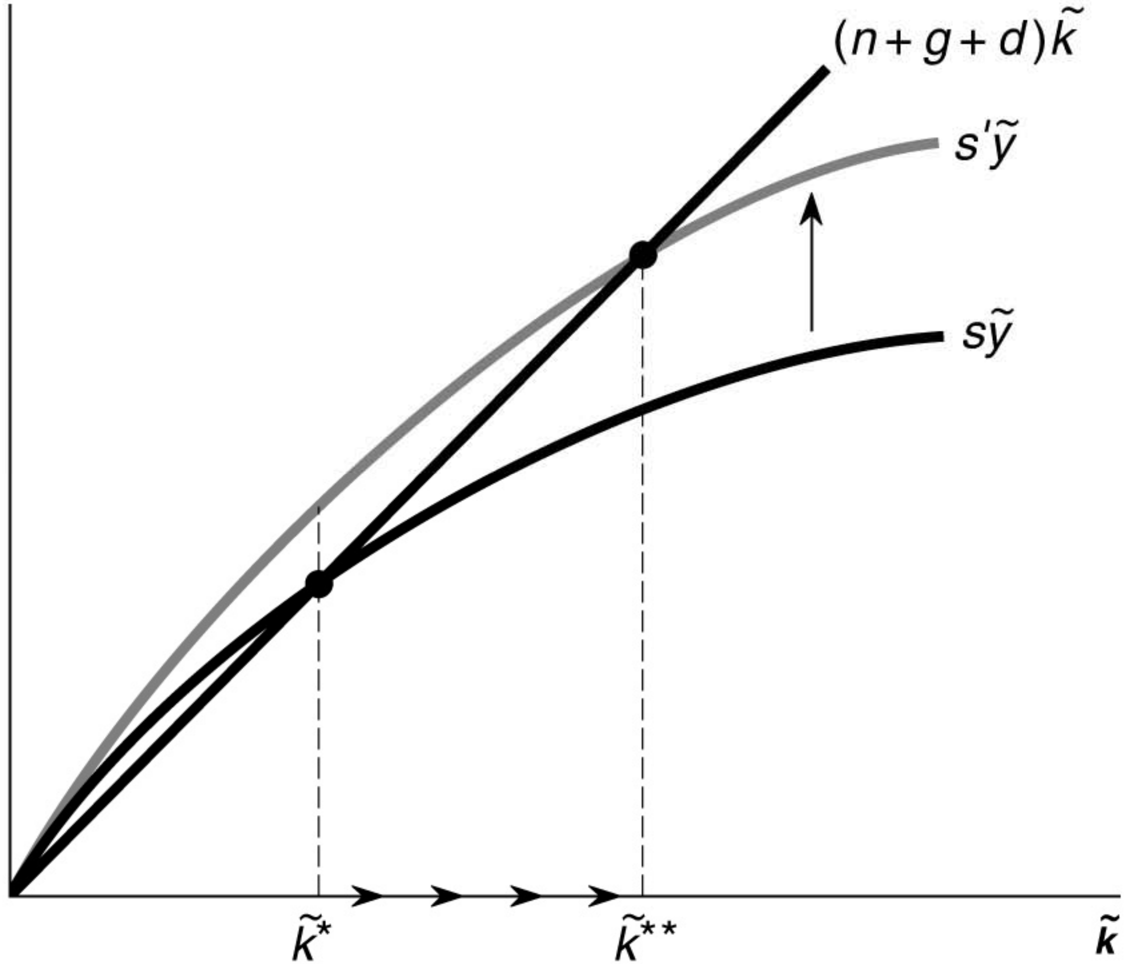

FIGURE 2.10 **AN INCREASE IN THE INVESTMENT RATE**

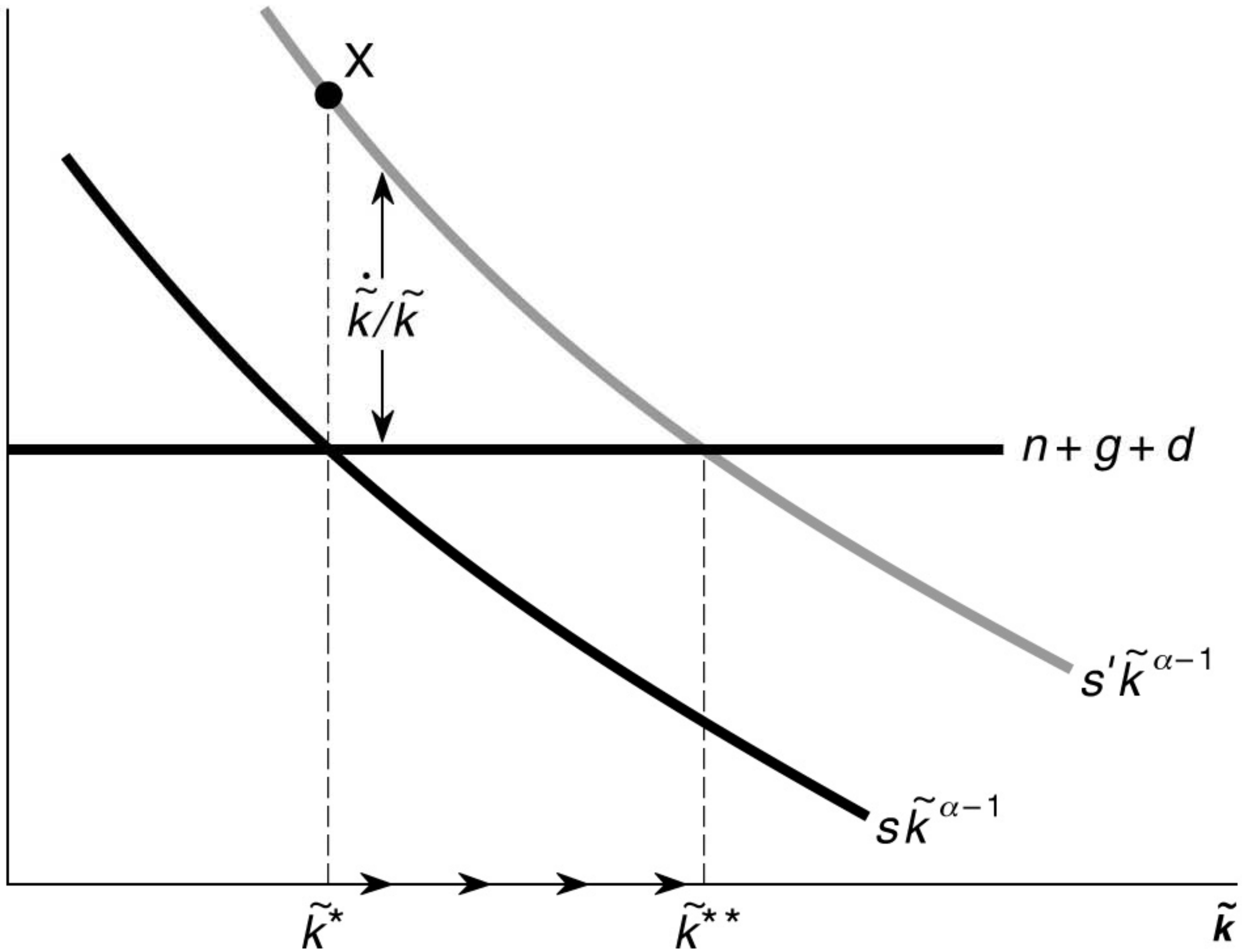

FIGURE 2.11 AN INCREASE IN THE INVESTMENT RATE: TRANSITION DYNAMICS

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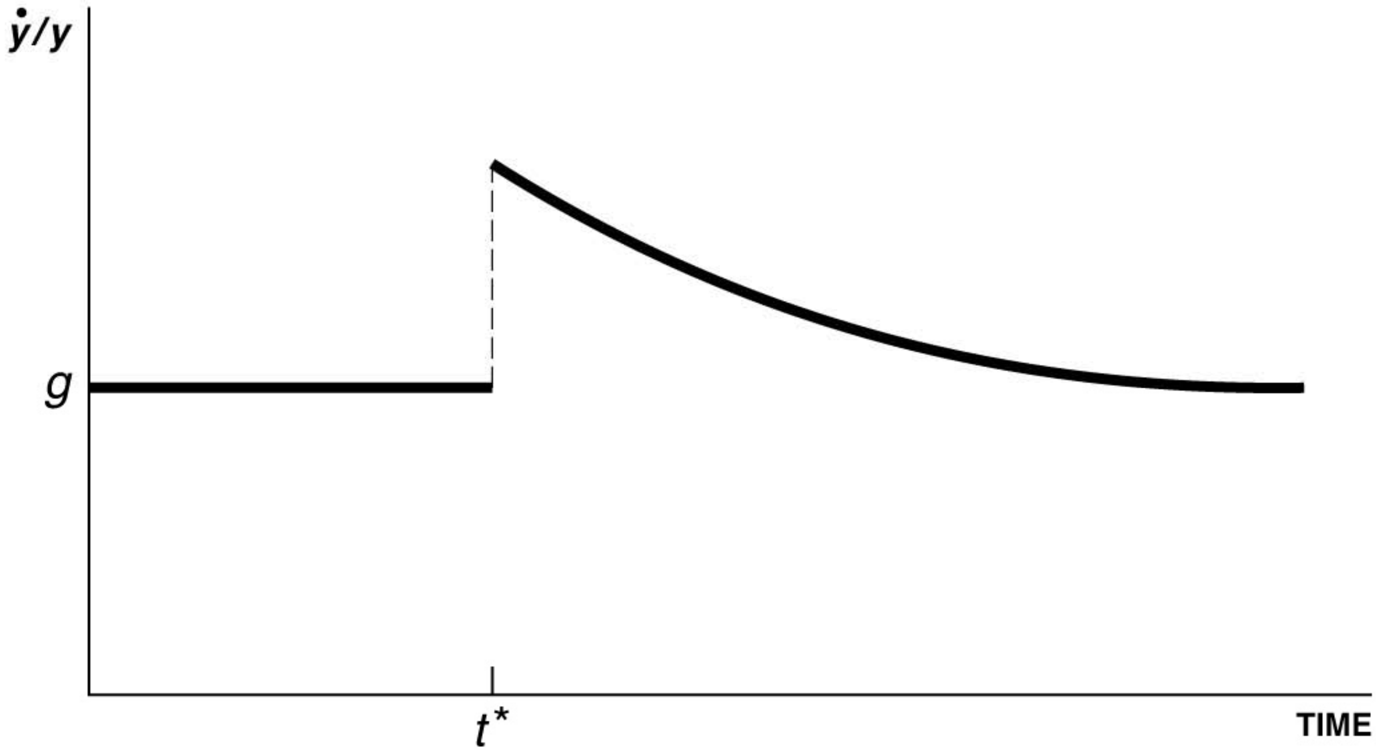

FIGURE 2.12 **THE EFFECT OF AN INCREASE IN INVESTMENT ON GROWTH**

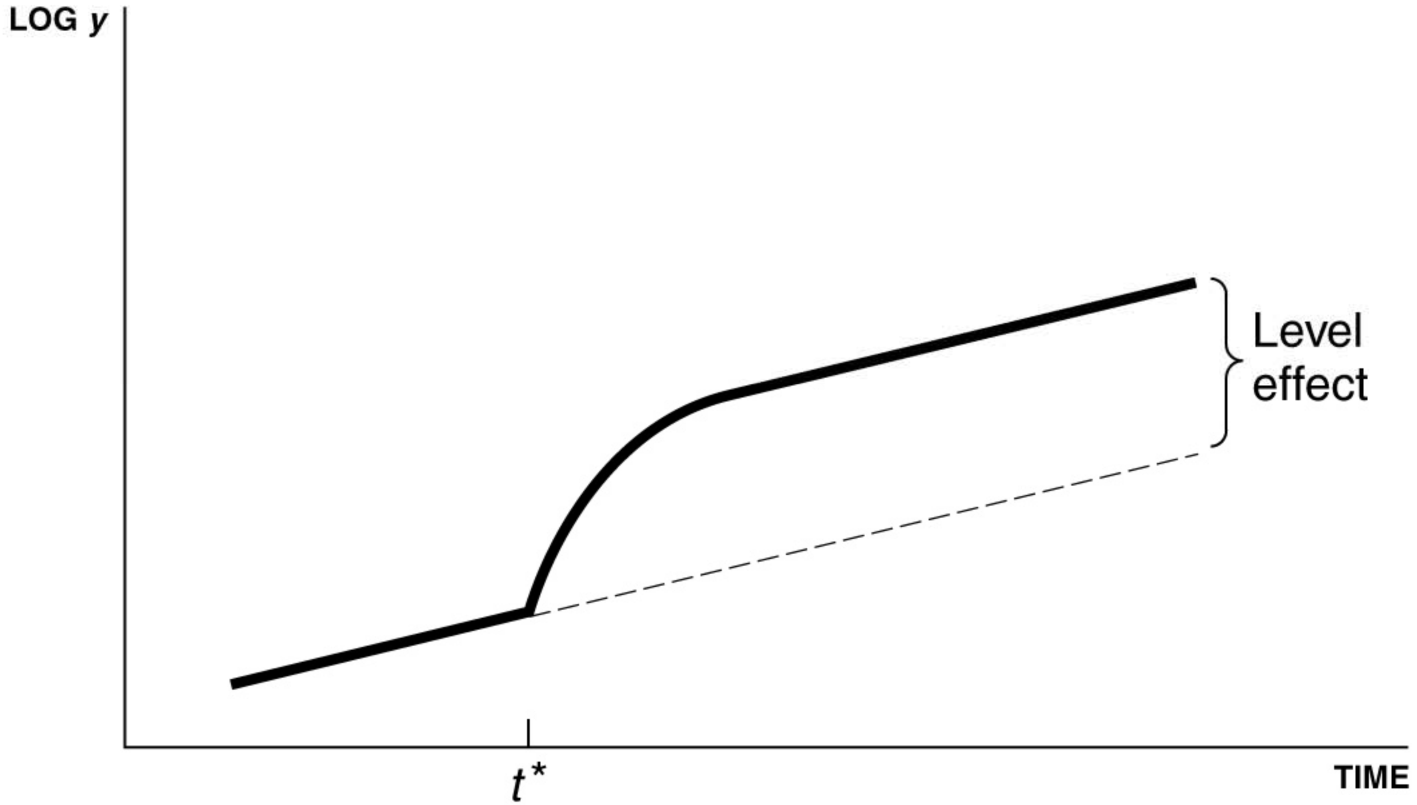

FIGURE 2.13 THE EFFECT OF AN INCREASE IN INVESTMENT ON y

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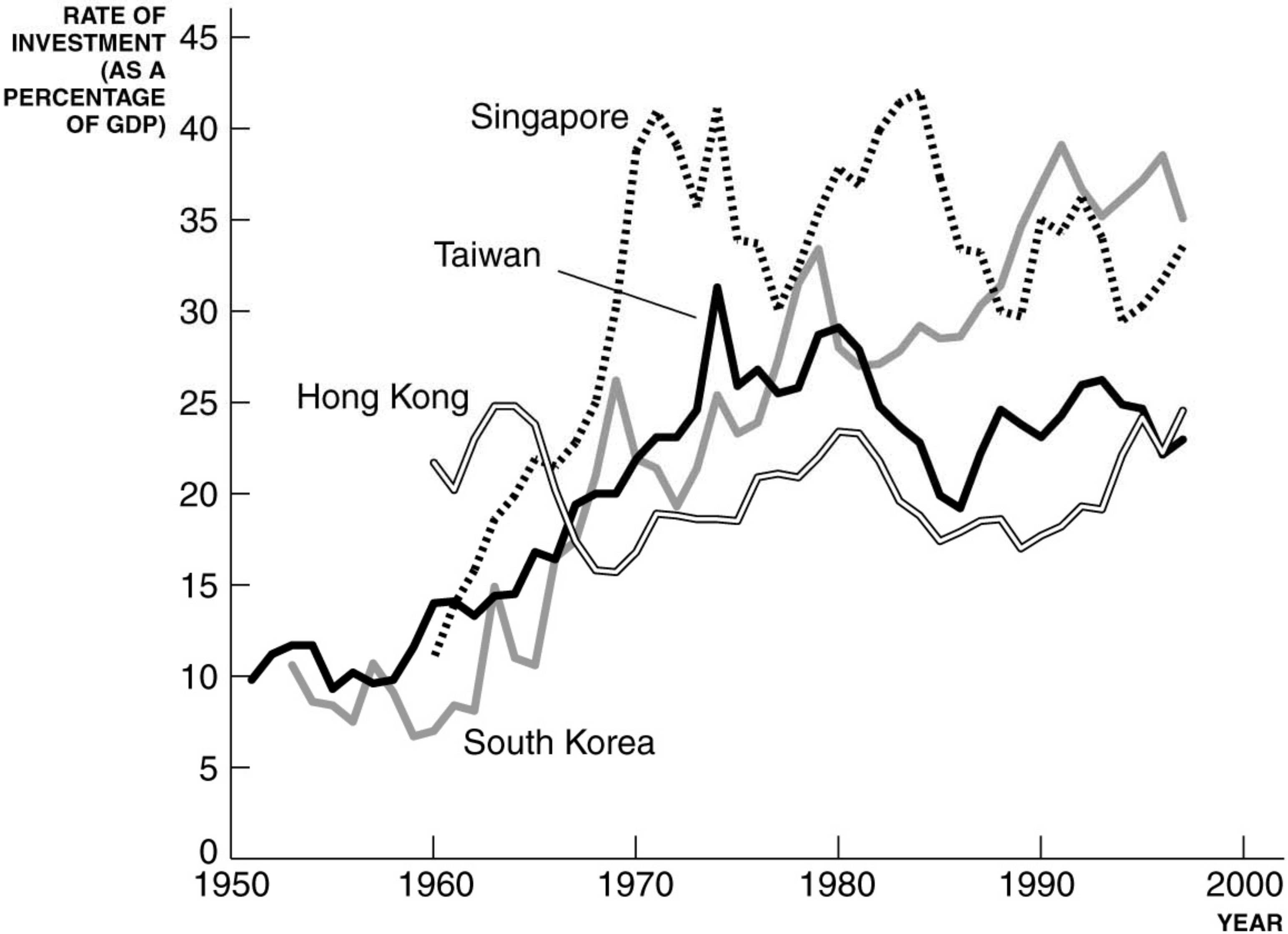

FIGURE 2.14 INVESTMENT RATES IN SOME NEWLY INDUSTRIALIZING ECONOMIES

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- What difference does technological progress bring?

The model now generates the growth in output per capita, and so can be mapped to the data better.

The disadvantage is that this growth is driven *exogenously*.

Growth accounting

Define $Y(t) = F(K(t), A(t)L(t)) = B(t)F(K(t), L(t))$. For example, if $Y(t) = K(t)^{\alpha}(A(t)L(t))^{1-\alpha}$, then $Y(t) = A(t)^{1-\alpha}K(t)^{\alpha}L(t)^{1-\alpha} = B(t)K(t)^{\alpha}L(t)^{1-\alpha}$, where $B(t) \equiv A(t)^{1-\alpha}$. Then,

$$\frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{B}(t)}{B(t)} + \alpha\frac{\dot{K}(t)}{K(t)} + (1-\alpha)\frac{\dot{L}(t)}{L(t)}$$

$$\frac{\dot{Y}(t)}{Y(t)} - \frac{\dot{L}(t)}{L(t)} = \frac{\dot{B}(t)}{B(t)} + \alpha\left(\frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)}\right)$$

$$\frac{\dot{y}(t)}{y(t)} = \frac{\dot{B}(t)}{B(t)} + \alpha\frac{\dot{k}(t)}{k(t)},$$

where $y(t) = \frac{Y(t)}{L(t)}$, and $k(t) = \frac{K(t)}{L(t)}$.

The contribution of the growth rate in $B(t)$ towards the observed growth rate in output per capita is called the "residual," or a "measure of our ignorance."

TABLE 2.1 GROWTH ACCOUNTING FOR THE UNITED STATES

	1948–98	48–73	73–79	79–90	90–95	95–98
Output per hour	2.5	3.3	1.3	1.6	1.5	2.5
Contributions from:						
Capital per hour worked	0.8	1.0	0.7	0.7	0.5	0.8
Information technology	0.3	0.1	0.3	0.5	0.4	0.8
Other capital services	0.6	0.9	0.5	0.3	0.1	0.0
Labor composition	0.2	0.2	0.0	0.3	0.4	0.3
Multifactor productivity	1.4	2.1	0.6	0.5	0.6	1.4

SOURCE: Bureau of Labor Statistics (2000).

Note: The table reports average annual growth rates for the private business sector. "Information technology" refers to information processing equipment and software.