Intermediate Macroeconomic Theory II, Winter 2011
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Solutions to Problem set 1

1. (**18 points**) Let the economy's production function be $Y = 4K^{1/2}(EL)^{1/2}$.
Households save 20% of their income;
population growth, n, is equal to 2%;
the depreciation rate, δ, is equal to 1%;
the growth rate in the efficiency of labor, g, is 1%.

   (a) (**2 points**) Show that the aggregate production function is constant returns to scale in K and L (**1 point**), and express the production function in *per-effective-worker* terms (**1 point**).

   (b) (**1 point**) Is production function increasing/constant/decreasing returns to scale in 3 factors of production, K, E, and L? *Show* how you arrived at the conclusion.

   (c) (**3 points**) Calculate the steady state level of
   capital *per effective worker* (**1 point**)
   output per *effective worker* (**1 point**)
   and consumption *per effective worker* (**1 point**).

   (d) (**1 point**) If you were a social planner who maximizes consumption per worker in the economy, what savings rate would you choose? (You *need not* show your calculations here if you see the answer.)

   (e) (**2 points**) Find the golden rule level of capital per effective worker (**1 point**) and the corresponding output per effective worker in the golden rule steady state (**1 point**).

   (f) (**2 points**) Assume the government can tax proportionally capital and wage incomes at the rate τ and is reinvesting the tax proceedings into capital. Find the tax rate τ that will deliver the golden level of capital per effective worker you have found immediately above (continue assuming that the savings rate is 20%).

   (g) (**2 points**) Assume now instead that a donor outside of the economy is willing to make a gift of capital to the economy so that it reaches the golden rule steady-state. How much capital would the donor need to bring into the economy relative to what it has when the steady state is defined by the savings rate of 20%.

   (h) (**1 point**) What is the growth rate of total output in the steady state (on a balanced growth path, to be precise)?

   (i) (**1 point**) What is the growth rate of the real wage in the economy? What is the growth rate of the real interest rate?

   (j) (**1 point**) What is the share of capital and labor costs in total income?

(k) (**2 points**) Assume the economy is on a balanced growth path. Let the production function be $Y = BK^{1/2}L^{1/2}$, where $B = 4E^{1/2}$, and B is the total factor productivity. What is the contribution of the total factor productivity towards the growth in total output? That is, calculate $\frac{\Delta B/B}{\Delta Y/Y}$.

Answer:

(a) $F(zK, zL) = 4(zK)^{\frac{1}{2}}(EzL)^{\frac{1}{2}} = z^{\frac{1}{2}}z^{\frac{1}{2}}\underbrace{4K^{\frac{1}{2}}(EL)^{\frac{1}{2}}}_{=Y} = zY$—raising utilization of K and L by the factor z magnifies output produced by the factor z. Define $z = \frac{1}{EL}$. Then, $\frac{Y}{EL} = F\left(\frac{K}{EL}, \frac{EL}{EL}\right) = 4\left(\frac{K}{EL}\right)^{\frac{1}{2}}\left(\frac{EL}{EL}\right)^{\frac{1}{2}}$. Thus, $y_{pew} = 4k_{pew}^{1/2}$, where $y_{pew} = \frac{Y}{EL}$ and $k_{pew} = \frac{K}{EL}$.

(b) Production function is increasing returns to scale in K, E and L: $F(zK, zEzL) = 4(zK)^{1/2}(zEzL)^{1/2} = z^{3/2}\underbrace{4K^{1/2}(EL)^{1/2}}_{=Y} > zY$. Thus, doubling the utilization of K, E and L more than doubles output, it magnifies output by the factor $2^{3/2}$.

(c) In steady state, $\Delta k_{pew} = s4k_{pew}^{1/2} - (n + \delta + g)k_{pew} = 0$. It follows that $k_{pew}^* = \left(\frac{4s}{n+g+\delta}\right)^{\frac{1}{1-1/2}} = \left(\frac{4s}{n+g+\delta}\right)^2 = \left(\frac{4\times 0.2}{0.04}\right)^2 = 400$.
$y_{pew}^* = 4\left(\frac{4\times 0.2}{0.04}\right) = 80$.
$c_{pew}^* = (1 - s)y_{pew}^* = 0.8 \times 80 = 64$.

(d) You should set the savings rate equal to the golden rule savings rate of 50%.

(e) At the golden rule steady state, $s = \alpha$, and so $k_{gold}^* = \left(\frac{4\alpha}{n+g+\delta}\right)^2 = \left(\frac{4\times 0.50}{0.04}\right)^2 = 2500$.
$y_{gold}^* = 4\left(\frac{4\times 0.50}{0.04}\right) = 200$.
In the golden rule steady state, capital intensity (i.e., capital per effective worker) is higher compared with the capital intensity in the steady state with $s = 0.20$.

(f) From our homework, we know that the steady-state level of capital per effective worker in the economy with the tax rate τ is obtained when $[s(1-\tau) + \tau]\, 4k_{pew}^{1/2} = (n + g + \delta)k_{pew}$. This happens when

$$k_{pew}^* = \left(\frac{4\,[s(1-\tau) + \tau]}{n + g + \delta}\right)^{\frac{1}{1-\alpha}}.$$

This level of capital per effective worker will be exactly equal to 2,500 when

$$4\,[s(1-\tau) + \tau] = 4 \times \alpha = 4 \times 0.5,$$

or when

$$\tau = \frac{0.5 - 0.2}{1 - 0.2} = 0.3/0.8 = 3/8 = 37.5\%.$$

(g) Denote the time the transfer happens as time 0. When the economy's savings rate is 20% its level of capital per effective worker in the steady state, $k^*_{pew} = \frac{K_0}{E_0 L_0}$, is 400. In the golden rule steady-state, capital per effective worker is $\frac{K'_0}{E_0 L_0} = 2500$. Thus, the donor needs to transfer $100(2500/400 - 1) = 525\%$ of the existing stock of capital.

(h) The balanced growth rate in total output is $n + g = 3\%$.

(i) The growth rate in the real wage will be equal to $\frac{\Delta y_{pew}}{y^*_{pew}} + \frac{\Delta E}{E} = 0 + g = 0.01$, or 1%. The real interest rate is constant: its growth rate is 0%.

(j) The share of capital costs in total income is 1/2, and the share of labor costs in total income is 1/2.

(k) On a balanced growth path, the growth rate of total output is equal to $n + g$=3%. The growth rate of B is $\frac{\Delta B}{B} = \frac{1}{2}\frac{\Delta E}{E} = \frac{1}{2}g$. Thus, $\frac{\Delta B/B}{\Delta Y/Y} = \frac{0.5\times 0.01}{0.03} \approx$ 17%.

2. (**10 points**) Consider the Permanent Income Hypothesis we studied in class. Preferences are quadratic, $u(c_t) = -\frac{1}{2}(\bar{c} - c_t)^2$; planning horizon is infinite; $\beta(1+r) = 1$; income stream is known as of time 0. Consider two individuals, X and Y. Individual X's income starts at 121 at time 0, drops to 110 at time 1 and stays at 110 thereafter, while individual Y's income starts at 110 at time 0, grows to 121 at time 1 and stays at 121 thereafter. The real interest rate, r, equals 10%.

(a) (**3 points**) Write down the lifetime budget constraint, and find the present discounted value of individual X's and Y's incomes.

(b) (**3 points**) Write down two optimality Euler conditions relating optimal consumption levels in periods 0 and 1, and optimal consumption levels in periods 1 and 2, assuming both X and Y are unconstrained in their borrowing.

Find optimal consumption levels for each period ($t = 0, 1, 2, 3, \ldots, \infty$) for both individuals assuming that they are free to save and borrow at the interest r.

(c) (**4 points**) Assume that individuals are precluded from borrowing. Show how this influences the path of consumption for both individuals.

Answer:

(a) The lifetime budget constraint relates the present discounted sum of consumption to the present discounted sum of incomes:

$$c_0 + \frac{c_1}{1+r} + \frac{c_2}{(1+r)^2} + \frac{c_3}{(1+r)^3} + \ldots = y_0 + \frac{y_1}{1+r} + \frac{y_2}{(1+r)^2} + \frac{y_3}{(1+r)^3} + \ldots$$

Individual X's present discounted value of incomes is

$$121 + \frac{110}{1+r} + \frac{110}{(1+r)^2} + \frac{110}{(1+r)^3} + \ldots = 121 + \frac{1}{1+r}\frac{110}{1 - \frac{1}{1+r}} = 1,221.$$

Individual Y's present discounted value of incomes is

$$110 + \frac{121}{1+r} + \frac{121}{(1+r)^2} + \frac{121}{(1+r)^3} + \ldots = 110 + \frac{1}{1+r}\frac{121}{1 - \frac{1}{1+r}} = 110 + \frac{121}{r} = 1,320.$$

(b) The optimality conditions are:

$$\bar{c} - c_0^* = \beta(1+r)\left(\bar{c} - c_1^*\right)$$

$$\bar{c} - c_1^* = \beta(1+r)\left(\bar{c} - c_2^*\right).$$

Since $\beta(1+r) = 1$ by assumption, it follows that the unconstrained consumption plan for both individuals is $c_0^* = c_1^* = c_2^* = c_3^* = \ldots$—perfect consumption smoothing.
Optimal consumption for individual X equals

$$\frac{r}{1+r}\left(121 + \frac{110}{r}\right) = 11 + 100 = 111.$$

Optimal consumption for individual Y equals

$$\frac{r}{1+r}\left(110 + \frac{121}{r}\right) = 10 + 110 = 120.$$

(c) Individual X's consumption path wouldn't change. She will spread her period 0 savings of 10 across her entire lifetime, consuming the interest each period.

Individual Y will not be able to set her consumption to the desired (unconstrained) level of 120, and she would have to set it to her endowment in period 0 equal to 110. When she arrives at period 1, her present discounted sum of incomes equals

$$121 + \frac{121}{1+r} + \frac{121}{(1+r)^2} + \frac{121}{(1+r)^3} + \ldots = 121\frac{1+r}{r},$$

and her new estimate of the permanent income equals

$$\frac{r}{1+r}121\frac{1+r}{r} = 121,$$

enabling her to set consumption to 121 in periods 1, 2, 3, and so on.