Intermediate Macroeconomic Theory II, Winter 2011
Instructor: Dmytro Hryshko
Solutions to the Midterm Exam

General Notes: **Bonus** question is optional. Good luck and have a nice reading week!

1. (**13 points**) Assume the Solow economy without technological progress. Let the economy's production function be $Y = \frac{1}{2}K^{1/2}L^{1/2}$.
   Households save 60% of their income;
   population growth, n, is equal to 2%;
   the depreciation rate, δ, is equal to 1%.

   (a) (**2 points**) Show that the aggregate production function is constant returns to scale in K and L (**1 point**), and express the production function in *per-worker* terms (**1 point**).

   (b) (**1 point**) Is production function increasing/constant/decreasing returns to scale in 1 factor of production, K? *Show* how you arrived at the conclusion.

   (c) (**3 points**) Calculate the steady state level of
   capital *per worker* (**1 point**)
   output per *worker* (**1 point**)
   and consumption *per worker* (**1 point**).

   (d) (**1 point**) If you were a social planner who maximizes consumption per worker in the economy, what savings rate would you choose? (You *need not* show your calculations here if you see the answer.)

   (e) (**2 points**) Find the golden rule level of capital per worker (**1 point**) and the corresponding output per worker in the golden rule steady state (**1 point**).

   (f) (**1 point**) Calculate the ratio of aggregate capital, K, to aggregate income, Y, in the steady state.

   (g) (**1 point**) Calculate the real interest rate in the steady state.

   (h) (**1 point**) What is the growth rate of total output and output per worker in the steady state?

   (i) (**1 point**) If you wanted to change the model so that it delivers a sustained growth rate of real wages over time, what would you introduce into the model?

   Answer:

   (a) $F(zK, zL) = \frac{1}{2}(zK)^{\frac{1}{2}}(zL)^{\frac{1}{2}} = z^{\frac{1}{2}}z^{\frac{1}{2}}\underbrace{\frac{1}{2}K^{\frac{1}{2}}L^{\frac{1}{2}}}_{=Y} = zY$—raising utilization of K and L by the factor z magnifies output produced by the factor z. Define $z = \frac{1}{L}$. Then, $\frac{Y}{L} = F\left(\frac{K}{L}, \frac{L}{L}\right) = \frac{1}{2}\left(\frac{K}{L}\right)^{\frac{1}{2}}\left(\frac{L}{L}\right)^{\frac{1}{2}}$. Thus, $y_{pw} = \frac{1}{2}k_{pw}^{1/2}$, where $y_{pw} = \frac{Y}{L}$ and $k_{pw} = \frac{K}{L}$.

(b) Production function is decreasing returns to scale in just K: $F(zK, L) = \frac{1}{2}(zK)^{1/2} L^{1/2} = z^{1/2} \underbrace{\frac{1}{2} K^{1/2} L^{1/2}}_{=Y} < zY$. Thus, doubling utilization of just K, and leaving utilization of L unaffected less than doubles output, it magnifies output by the factor $2^{1/2}$.

(c) In steady state, $\Delta k_{pw} = s \times 0.5 \times k_{pw}^{1/2} - (n + \delta) k_{pw} = 0$. It follows that $k^*_{pw} = \left(\frac{0.5s}{n+\delta}\right)^{\frac{1}{1-1/2}} = \left(\frac{0.5s}{n+\delta}\right)^2 = \left(\frac{0.5\times 0.6}{0.03}\right)^2 = 100$.
$y^*_{pew} = 0.5\sqrt{100} = 5$.
$c^*_{pew} = (1-s) y^*_{pew} = 0.4 \times 5 = 2$.

(d) You should set the savings rate equal to the golden rule savings rate of 50%.

(e) At the golden rule steady state, $s = \alpha$, and so $k^*_{gold} = \left(\frac{0.5\alpha}{n+\delta}\right)^2 = \left(\frac{0.5\times 0.5}{0.03}\right)^2 \approx 69.4$.
$y^*_{gold} = 0.5\left(\frac{0.5\times 0.5}{0.03}\right) \approx 4.17$.

(f) Note that $\frac{K}{Y} = \frac{K/L}{Y/L} = \frac{k_{pw}}{y_{pw}}$. In the steady state with the savings rate of 60%, $K/Y = 100/5 = 20$; in the golden rule steady state, $\frac{K}{Y} = \frac{\alpha}{n+\delta} \approx 16.7$. (**Any of those two answers should be fine.**)

(g) The real interest rate is $r = \text{MPK} - \delta = 0.5\frac{Y}{K} - \delta = 0.5\frac{y_{pw}}{k_{pw}} - \delta = 0.5 * 0.05 - 0.01 = 0.015$ or 1.5%. In the golden rule steady state, $r = n = 0.02$, or 2%. (**Any of those two answers should be fine.**) Clearly, the real interest rate is higher in the less capital-intensive golden rule steady state.

(h) The growth rate of output per worker is 0; the growth rate of total output is $n = 0.02$, or 2%.

(i) Should introduce technological progress (labor efficiency growing at a positive and constant rate).

2. (**7 points**) Consider the Solow economy without technological progress and the aggregate production function $Y = 10K^{1/2}L^{1/2}$; the economy's current saving rate is 20%, population doesn't grow over time and the depreciation rate equals $\delta > 0$. (You don't need the exact number for δ to answer this question.)

(a) (**4 points**) Assume a beneficial donor proposes households to bring in capital into their economy under the condition that households permanently change their savings rate to 50% once the necessary capital is installed. What is the percentage of existing capital the donor should bring into the economy so that per capita consumption is maximized? Draw a graph that shows the evolution of consumption per capita under the proposed plan. (Time on the horizontal axis and consumption per capita on the vertical axis.)

(b) (**2 points**) Now instead assume a council of households is arranged to decide on the optimal savings rate in the economy—the one maximizing consumption of the average person in the economy. Draw a graph that shows the evolution of consumption per capita once the optimal savings rate is implemented in the economy permanently. (Time on the horizontal axis and consumption per capita on the vertical axis.)

(c) (**1 point**) Under what scenario, (2a) or (2b), are households better off, and why?

Answer:

(a) The steady-state value of capital per worker equals $\left(\frac{10\times s}{\delta}\right)^2$. The donor's plan would ensure that the economy would immediately jump to the golden-rule value of capital per worker and stay there forever as households would be able to sustain it by increasing their savings rate permanently to 50%. The ratio of capital in the golden rule steady state to the capital in the steady state with the 20% savings rate is $\left(\frac{\alpha}{s}\right)^2 = 2.5^2 = 6.25$. In percentage terms, the donor needs to install $100 \times (6.25 - 1) = 525\%$ of the existing capital. See my graph on the last page.

(b) The economy should adopt the savings rate of 50% but will have to sacrifice the current (and some periods onward) consumption by investing more. In the very long, consumption per capita will be higher and at the same value as in the previous scenario. See my graph on the last page.

(c) Households will be better off under the first scenario as they don't need to sacrifice anything to reach a permanently higher level of consumption. After all, you would always be better off if you're getting something for free!

3. (**15 points**) Consider the following two-period problem of consumption. At time 0 and time 1 preferences are quadratic, $u(c_t) = -\frac{1}{2}(c_t - \bar{c})^2$ (t is either zero or 1); $\beta(1+r) = 1$; income stream is known as of time 0. Consider two individuals, X and Y. Individual X's income starts at 121 at time 0, but drops to 110 at time 1, while individual Y's income starts at 110 at time 0, but grows to 121 at time 1. The real interest rate, r, equals 10%.

(a) (**2 points**) What does $\bar{c}$ stand for in the utility function? Briefly elaborate (that is, do not simply state the label but give your interpretation.)

(b) (**2 points**) Write down the lifetime budget constraint, and find the present discounted value of individual X's and Y's incomes.

(c) (**3 points**) Write down the optimality Euler condition relating optimal consumption levels in periods 0 and 1, assuming both X and Y are unconstrained in their borrowing. Make a brief comment on the assumption $\beta(1+r) = 1$ that we're using; what does it deliver?

(d) (**4 points**) Find the optimal consumption levels for each period for both individuals assuming that they are free to save and borrow at the interest r.

(e) (**4 points**) Assume that individuals are precluded from borrowing. Show how this influences the path of consumption for both individuals.

Answer:

(a) It stands for the "bliss" consumption level; it is the level of consumption that gives the highest possible utility and therefore well-being (the utility index will be zero is consumption is set to $\bar{c}$ in every period, otherwise it will be negative). **Some of you wrote that it's some sort of an average consumption, this is clearly wrong.**

(b)

$$c_0 + \frac{c_1}{1+r} = y_0 + \frac{y_1}{1+r}.$$

The present discounted value of X's income is: $121 + \frac{110}{1+0.1} = 221$.

The present discounted value of Y's income is: $110 + \frac{121}{1+0.1} = 220$.

(c) The optimality condition, for this utility function, is

$$c_0^* - \bar{c} = \underbrace{\beta(1+r)}_{=1}(c_1^* - \bar{c}) = c_1^* - \bar{c},$$

which reduces to

$$c_0^* = c_1^*.$$

The assumption $\beta(1+r)$ ensures that the optimal consumption plan is to perfectly smooth consumption over time; moreover, for these preferences, we wouldn't be able to solve for the consumption levels as $\bar{c}$ will not get eliminated from the optimality condition.

(d) The optimal unconstrained consumption in both periods will be

$$c^* = \frac{1+r}{2+r} \times \text{PDVY},$$

where PDVY is the present discounted value of lifetime incomes.

Thus, individual X's optimal consumption is $\frac{1.1}{2.1}220 \approx 115.24$.

Individual Y's optimal consumption is $\frac{1.1}{2.1}221 \approx 115.76$.

(e) For X, the path does not change as she is optimally a saver.

Y's consumption in period 0 will be 110, and 121 in period 1—she will set her consumption to the endowments in each period.

4. (**Bonus 5 points**) Consider the Permanent Income Hypothesis we studied in class. Preferences are quadratic, $u(c_t) = -\frac{1}{2}(c_t - \bar{c})^2$; planning horizon is infinite; $\beta(1+r) = 1$; income stream is known as of time 0. Consider an individual X whose income starts at 100 at time 0 and grows at the net rate 2% each year, $g_X = 0.02$. The real interest rate, r, equals 4%.

(a) (**2 points**) Find the optimal consumption for the individual X assuming that she is not borrowing constrained.

(b) (**3 points**) Assume that individual X cannot borrow but can save at the interest r equal 4% up to period 10; in period 10, a financial institution observing individual X is convinced in the individual's growing earnings capacity and allows her to borrow at the rate 4% from period 10 and thereafter. What is X's optimal level of consumption in all periods under this scenario?

Answer:

(a) The present discounted value of X's incomes, S_X equals

$$S_X = 100\left(1 + \frac{1+g}{1+r} + \left(\frac{1+g}{1+r}\right)^2 + \left(\frac{1+g}{1+r}\right)^3 + \left(\frac{1+g}{1+r}\right)^4 + \dots\right)$$
$$= \frac{100}{1 - \frac{1+g}{1+r}} = \frac{100(1+r)}{r-g} = \frac{100(1+0.04)}{0.04 - 0.02} = 5,200.$$

The optimality/Euler equations for these preferences imply that consumption is constant in all periods given the consumer is not borrowing constrained; denote this optimal level of consumption as c^*. Using my note,

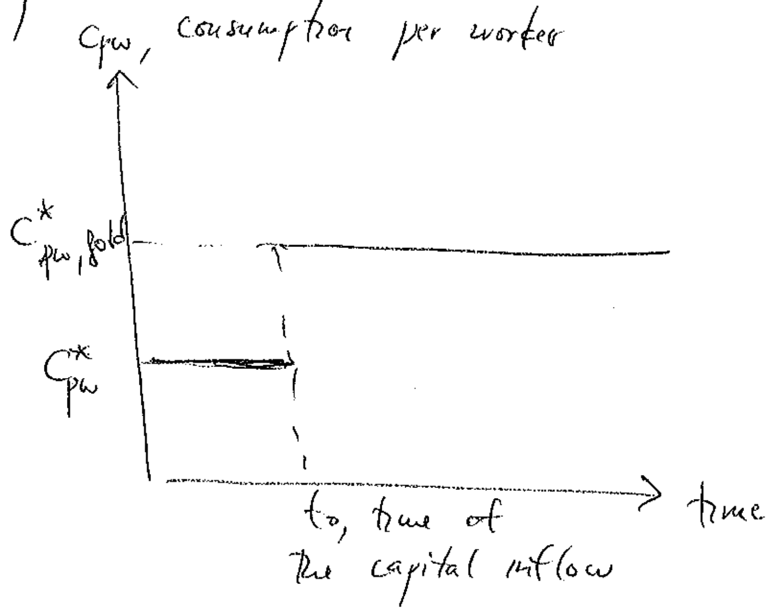
$$c_X^* = \frac{r}{1+r}S_X = \frac{0.04}{1.04} \times 5200 = 200,$$

where c_X^* is the optimal consumption level of individual X.

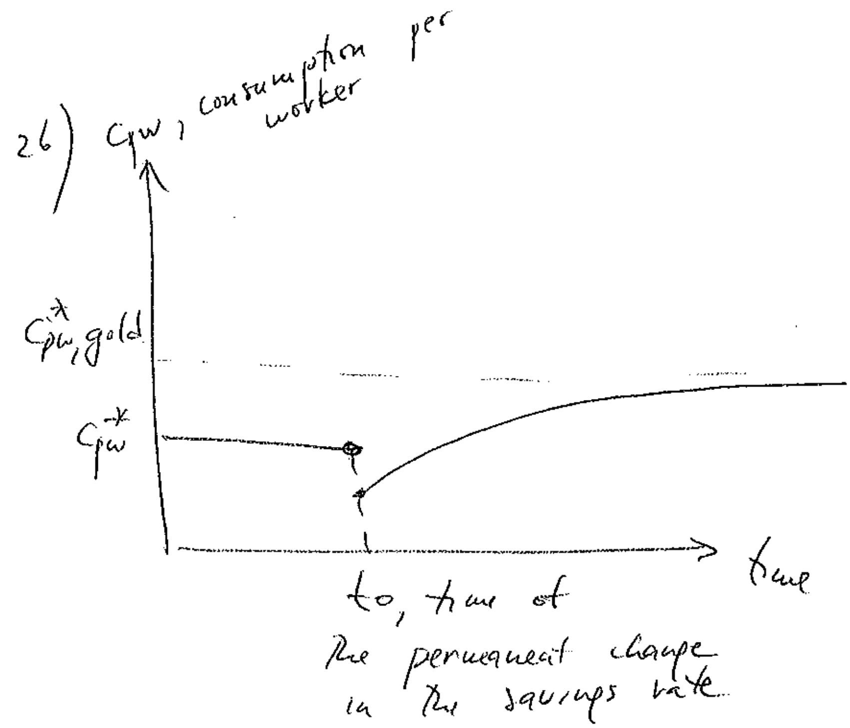
(b) For periods 0 to 9, X will set her consumption to her endowment in the respective period. After period 9, her consumption will be set to her permanent income and stay constant at periods 10, 11, 12, etc.:

$$c^* = \frac{r}{1+r}\left(100(1+g)^{10} + 100\frac{(1+g)^{11}}{1+r} + +100\frac{(1+g)^{12}}{(1+r)^2} + \dots\right)$$
$$= \frac{r}{1+r}100(1+g)^{10}\left(1 + \frac{1+g}{1+r} + \left(\frac{1+g}{1+r}\right)^2 + \left(\frac{1+g}{1+r}\right)^3 + \dots\right)$$
$$= \frac{r}{1+r}100(1+g)^{10}\frac{1+r}{r-g} = \frac{r}{r-g}100(1+g)^{10}$$
$$= \frac{0.04}{0.02}100(1.02)^{10} \approx 243.80$$

2a)


Cpw, consumption per worker
C*pw,gold
C*pw
to, time of
the capital inflow
time

2b)


Cpw, consumption per worker
C*pw,gold
C*pw
to, time of
the permanent change
in the savings rate
time