Intermediate Macroeconomic Theory II, Winter 2010
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Solutions to the Midterm Exam

1. (**21 points**) Let the economy's production function be $Y = 4K^{1/4}(EL)^{3/4}$.
Households save 40% of their income;
population growth, n, is equal to 1%;
the depreciation rate, δ, is equal to 2%;
the growth rate in the efficiency of labor, g, is 1%.

   (a) (**2 points**) Show that the aggregate production function is constant returns to scale in K and L (**1 point**), and express the production function in *per-effective-worker* terms (**1 point**).

   (b) (**1 point**) Is production function increasing/constant/decreasing returns to scale in 3 factors of production, K, E, and L? *Show* how you arrived at the conclusion.

   (c) (**3 points**) Calculate the steady state level of
   capital *per effective worker* (**1 point**)
   output per *effective worker* (**1 point**)
   and consumption *per effective worker* (**1 point**).

   (d) (**1 point**) If you were a social planner who maximizes consumption per worker in the economy, what savings rate would you choose? (You *need not* show your calculations here if you see the answer.)

   (e) (**2 points**) Find the golden rule level of capital per effective worker (**1 point**) and the corresponding output per effective worker in the golden rule steady state (**1 point**).

   (f) (**3 points**) Calculate the real interest rate under the economy's current savings rate (40%), when the economy is in the steady state. (**1 point**)
   Calculate the real interest rate when the economy is in the golden rule steady state. (**1 point**)
   If the interest rates differ, briefly argue why one is higher than another. (**1 point**)

   (g) (**1 point**) Calculate the real wage on a balanced growth path, when the economy's savings rate is 40%. (*Hint*: the real wage at time t will be a function of E at time t.)

   (h) (**1 point**) What is the growth rate of total output in the steady state (on a balanced growth path, to be precise)?

   (i) (**1 point**) What is the growth rate of the real wage in the economy? What is the growth rate of the real interest rate?

   (j) (**1 point**) What is the share of capital and labor costs in total income?

(k) (**2 points**) Assume the economy is on a balanced growth path. Let the production function be $Y = BK^{1/4}L^{3/4}$, where $B = 4E^{3/4}$, and B is the total factor productivity. What is the contribution of the total factor productivity towards the growth in total output? That is, calculate $\frac{\Delta B/B}{\Delta Y/Y}$.

(l) (**2 points**) Assume economy is on a balanced growth path, where s, n, g and δ are as before. Assume that at some time T savings rate is reduced permanently from 40% to 25%. Calculate the growth rate of real wages at time T. (**1 point**) What is the growth rate of real wages when the economy is on its new balanced growth path? (**1 point**)

(m) (**1 point**) Is the following statement true or not? For the just described economy, consumption per worker, as well as capital per worker and output per worker, will be higher in the steady state with the savings rate equal to 40% rather than 25%.

Answer:

(a) $F(zK, zL) = 4(zK)^{\frac{1}{4}}(EzL)^{\frac{3}{4}} = z^{\frac{1}{4}}z^{\frac{3}{4}}\underbrace{4K^{\frac{1}{4}}(EL)^{\frac{3}{4}}}_{=Y} = zY$—raising utilization of K and L by the factor z magnifies output produced by the factor z. Define $z = \frac{1}{EL}$. Then, $\frac{Y}{EL} = F\left(\frac{K}{EL}, \frac{EL}{EL}\right) = 4\left(\frac{K}{EL}\right)^{\frac{1}{4}}\left(\frac{EL}{EL}\right)^{\frac{3}{4}}$. Thus, $y_{pew} = 4k_{pew}^{1/4}$, where $y_{pew} = \frac{Y}{EL}$ and $k_{pew} = \frac{K}{EL}$.

(b) Production function is **increasing returns to scale in** K**,** E **and** L: $F(zK, zEzL) = 4\,(zK)^{1/4}\,(zEzL)^{3/4} = z^{7/4}\underbrace{4K^{1/4}(EL)^{3/4}}_{=Y} > zY$. Thus, doubling the utilization of K, E and L more than doubles output, it magnifies output by the factor $2^{7/4}$.

(c) In steady state, $\Delta k_{pew} = s4k_{pew}^{1/4} - (n + \delta + g)k_{pew} = 0$. It follows that
$k^*_{pew} = \left(\frac{4s}{n+g+\delta}\right)^{\frac{1}{1-1/4}} = \left(\frac{s}{n+g+\delta}\right)^{4/3} = \left(\frac{4\times0.4}{0.04}\right)^{4/3} \approx 136.80.$
$y^*_{pew} = 4\left(\frac{4\times0.4}{0.04}\right)^{1/3} \approx 13.68.$
$c^*_{pew} = (1 - s)y^*_{pew} = 0.6 \times 13.68 \approx 8.21.$

(d) You should set the savings rate equal to the golden rule savings rate of 25%.

(e) At the golden rule steady state, $s = \alpha$, and so
$k^*_{gold} = \left(\frac{4\alpha}{n+g+\delta}\right)^{4/3} = \left(\frac{4\times0.25}{0.04}\right)^{4/3} \approx 73.10.$
$y^*_{gold}{=}4\left(\frac{4\times0.25}{0.04}\right)^{1/3} \approx 11.70.$
You can verify (not required) that consumption per effective worker in the golden rule steady state is higher than in the steady state when $s = 40\%$ at $(1 - 0.25) \times 11.70 \approx 8.78$.
In the golden rule steady state, capital intensity (i.e., capital per effective worker) is lower compared with the capital intensity in the steady state with $s = 0.3$.

(f) The real interest rate is obtained from $r = MPK - \delta$. For the Cobb-Douglas constant returns to scale production function, $MPK = \frac{1}{4}\frac{Y}{K} = \frac{1}{4}\frac{y_{pew}}{k_{pew}}$. In the steady state, $\frac{y^*_{pew}}{k^*_{pew}} = \frac{4(4s/(n+g+\delta))^{1/3}}{(4s/(n+g+\delta))^{4/3}} = 4\frac{n+g+\delta}{4s}$.

Thus, $r^* = \frac{1}{4}\frac{n+g+\delta}{s} - \delta = 0.005$, or 0.5%.

In the golden rule steady state, $MPK = n+\delta+g$, and $r = MPK - \delta = n + g = 0.02$, or 2%.

The interest rate is definitely higher in the golden rule steady state because the return to a marginal unit of capital is higher in the golden rule steady state, as the capital is more scarce.

(g) The real wage is equal to the marginal product of labor $(1-\alpha)\frac{Y}{L} = (1-\alpha)y_{pew}E = 0.75 \times 13.68 \times E$. Notice that the real wage will change over time since the efficiency of labor is assumed to grow exogenously at the rate of 1%.

Thus, for any time t, $w(t) = 10.26E(0)(1+0.01)^t$, where $E(0)$ is the (initial) level of labor efficiency at time 0.

(h) Since $Y = y_{pew}EL$, $\frac{\Delta Y}{Y} = \frac{\Delta y_{pew}}{y_{pew}} + \frac{\Delta L}{L} + \frac{\Delta E}{E}$. In the steady state, y_{pew} is constant and not growing. Therefore, $\frac{\Delta Y}{Y} = n + g = 0.02 = 2\%$.

(i) **The growth rate in the real wage** will be equal to $\frac{\Delta y_{pew}}{y^*_{pew}} + \frac{\Delta E}{E} = 0 + g = 0.01$, or 1%. **The real interest rate is constant**: its growth rate is 0%.

(j) **The share of capital costs** in total income is $\alpha = 1/4$, and **the share of labor costs** in total income is $1 - \alpha = 3/4$.

(k) On a balanced growth path, the growth rate of total output is equal to $n + g$=2%. The growth rate of B is $\frac{\Delta B}{B} = \frac{3}{4}\frac{\Delta E}{E} = \frac{3}{4}g$.

Thus, $\frac{\Delta B/B}{\Delta Y/Y} = \frac{0.75\times0.01}{0.02} = 37.5\%$.

(l) The growth rate in the real wage will be equal to $\frac{\Delta y_{pew}}{y^*_{pew}} + \frac{\Delta E}{E} = \frac{\Delta y_{pew}}{y^*_{pew}} + 0.01 = \frac{1}{4}\frac{\Delta k_{pew}}{k^*_{pew}} + 0.01$. At time T, the change in capital per effective worker will be equal to $\frac{\Delta k_{pew}}{k^*_{pew}} = 0.25 * \frac{y^*_{pew}}{k^*_{pew}} - (n+g+\delta) = 0.25 * \frac{n+\delta+g}{0.4} - (n+g+\delta) = 0.04 * \left(\frac{0.25}{0.4} - 1\right) = -0.015$.

Thus, **the growth rate in the real wage at time** T is $-\frac{1}{4} * 0.015 + 0.01 = 0.00625$, or 0.625%.

When the economy evolves along its new balanced growth path, the growth rate in the real wage will be equal to 1%.

(m) **False**. Consumption per worker will be higher in the steady state with the savings rate equal 25% (golden-rule steady state), while capital per worker and output per worker will be lower in that steady state.

2. (**5 points**) Consider someone deciding how to allocate her consumption over two peri-

ods. She has utility function $U(C_1, C_2) = -0.5(C_1 - \overline{C})^2 - 0.5\beta(C_2 - \overline{C})^2$, $0 \leq \beta \leq 1$. β is called the time discount factor, and measures how one values current consumption versus future consumption, or, in other words, one's impatience (the lower the β, the more impatient you are, i.e., the more you value current consumption relative to future consumption); $\overline{C}$ is the "bliss" level of consumption. (*Hint*: for this utility function, $MU_1 = \overline{C} - C_1$.)
Assume that $\beta(1+r) = 1$; income in the first period is $Y_1 = 100$, income in the second period is $Y_2 = 200$; and the real interest rate, r, is equal to zero.

(a) (**2 points**) Solve for optimal consumption in each period. (**1 point**) Determine the amount of optimal savings/borrowing in the first period of life. (**1 point**)

(b) (**2 points**) Briefly describe your answer in terms of the Permanent Income Hypothesis.

(c) (**1 point**) Would your answer differ if we assumed that the agent may save at the interest r but is unable to borrow at all? If yes, what would be the optimal consumption in periods 1 and 2 in this case?

Answer:

(a) The optimality Euler equation, for this utility function, is $\overline{C} - C_1^* = \beta(1+r)(\overline{C} - C_2^*)$. Since we assumed that $\beta(1+r) = 1$, $C_1^* = C_2^*$. Utilizing the intertemporal budget constraint, $C_1 + C_2/(1+r) = Y_1 + Y_2/(1+r)$, or $C_1 + C_2 = Y_1 + Y_2$.
The **optimal consumption** is $C_1^* = C_2^* = 0.5(100 + 200) = 150$.
Optimal savings are $S_1^* = Y_1 - C_1^* = 100 - 150 = -50$.

(b) For the preferences specified in this problem and our assumption that $\beta(1+r) = 1$, consumer would desire to smooth consumption across periods, given she is not borrowing constrained. Thus, regardless of the fact that income is not even across periods, consumer is able to smooth her consumption perfectly by borrowing against higher future endowment.
In accordance with the PIH, consumer sets her consumption equal to the permanent income and smoothes her consumption if she is not "banned" from credit markets.

(c) The result would change since we showed that optimally consumer is a borrower.

If our consumer is prevented from borrowing, she will set her consumption equal to endowments in both periods $C_1 = Y_1 = 100$ and $C_2 = Y_2 = 200$.

3. (**5 points**)

Briefly comment the following statements, and show how you arrived at your conclusion.

(a) (**1 point**) The real interest rate is constant in the steady-state of the Solow economy with population growth but no technological growth, while it is growing on a balanced growth path of the economy with positive population and technological growth.

(b) (**1 point**) Ceteris paribus, we can predict that the price of capital and the real interest rate will go down if some part of the economy's capital stock gets destroyed, say, due to war.

(c) (**1 point**) Ceteris paribus, we can predict that the price of capital and the real interest rate will go down if there is an inflow of immigrants into the economy.

(d) (**1 point**) Consider the following situation. One receives some news that she will receive a large inheritance in 10 years with a very high likelihood. The Permanent Income Hypothesis predicts that the person will not act on this news now in terms of current consumption but rather wait for 10 years and then change her consumption if that inheritance is indeed received.

(e) (**1 point**) Consider the Solow economy without technological growth, in its steady state equilibrium. At some point in time, part of the economy's stock of capital is destroyed but otherwise the economy remains the same as before, that is the savings patterns are the same, population growth, depreciation rate, as well as the aggregate technology are the same. What will happen eventually to this economy?

Answer:

(a) **False**. The real interest rate can be written as $r = \alpha \frac{Y}{K} - \delta = \alpha \frac{y_{pw}}{k_{pw}} - \delta = \alpha \frac{y_{pew}}{k_{pew}} - \delta$. The last two equalities highlight the fact that the real interest rate is constant in the steady state of the economy with positive population but no technological growth (next-to-last equality) and on a balanced growth path of the economy with positive population and technological growth (last equality).

(b) **False**. The real interest rate is related to the marginal productivity of capital, and it will be higher for lower levels of capital by the law of diminishing returns. Intuitively, when a factor of production becomes scarce, it commands a higher price.

(c) **False**. With an inflow of immigrants, the productivity of capital goes up, and one unit of capital will command a higher price (again, you can think in terms of relative scarcity of labor and capital).

(d) **False**. The point of the PIH that one would want to smooth consumption and will react to the news about permanent income when the news is received, not when the actual change in resources happen. Thus, provided our consumer is not liquidity constrained, she will reset her consumption now to a higher value.

(e) The economy will eventually return to exactly the same levels of capital per worker, output per worker and consumption per worker. The effects of the shock are not permanent. Of course, there will be transitional dynamics.

On the impact of the shock, the capital-labor ratio will fall, the amount of output per worker produced will fall as well, the economy will start accumulating capital so that the capital-labor ratio as well as output per worker will eventually return to their pre-shock values.