Chapter 4: Economic Growth I (No Technological Growth)

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Determinants of well-being and economic growth

Previously we analyzed the determinants of output at a point in time.

1. Now we want to understand the economic dynamics, i.e. what are the factors behind the growth in GDP

2. What determines the economic well-being of the average person in the economy, or what are the factors behind the growth in GDP per worker

★ We shape our understanding using the Solow growth model
We start with a simplified variant of the Solow model.

It is a neoclassical model, where prices have no role in the determination of real quantities, like capital and output.

Assume for now there is no growth in working population and there is no technological growth. Later we’ll relax these assumptions.
Supply of Goods

- As before, supply of goods, $Y$ is determined by the production function in the economy, $F(K, L)$.

- Assume that production function is of the CRS type. That is: $F(zK, zL) = zF(K, L) = zY$.

- Let $z = \frac{1}{L}$. Then $F\left(\frac{K}{L}, \frac{L}{L}\right) = \frac{Y}{L}; F\left(\frac{K}{L}, 1\right) = \frac{Y}{L}$.

- Denote $y = \frac{Y}{L}; k = \frac{K}{L}$. I.e. lowercase letters denote ratios of respective variables to the (working) population. Let $F\left(\frac{K}{L}, 1\right) = f(k)$. 
The Marginal Product of Capital

- We can rewrite production function in per capita terms as:  
  \[ y = f(k). \]

- As usual, the marginal product of capital is measured by the slope of the production function at any given level of \( k \).

  \[ MPK = \frac{\Delta f(k)}{\Delta k} = f(k + 1) - f(k), \]

  We increment \( k \) by one unit so that \( \Delta k = 1 \).
Assume that economy is closed, \( NX = 0 \), and \( G = 0 \). Then,

\[ Y = C + I, \quad y = c + i. \]

This means that output is consumed by domestic households and purchased by investors to enlarge and replace their capital stock.

A key assumption of the Solow growth model: saving rate \( s \)—the fraction of output/income saved from additional dollar—is exogenously given (fixed).

Then, \( c = (1 - s)y = (1 - s)f(k) \) and \( y = (1 - s)y + i \), or \( i = sy = sf(k) \).
The Law of Motion of Capital

- Now we know that consumption per worker, output per worker, and investment per worker are functions of capital per worker only. This leads us to the question:

- How is the capital per worker determined? New capital is added each period by adding investment to the old stock of capital, and a portion of old capital wears off in the production process which leads to a lower capital stock. The process of ‘losing’ capital in the process of production is called depreciation.

- Let depreciation rate be $\delta$. E.g., $\delta = 0.1$ means that each year 10% of capital per worker is ‘lost’/wears off in production process.
Steady-State Level of Capital

- Summarizing, \( k_{t+1} = k_t + i_t - \delta k_t \). Or, \( \Delta k_{t+1} = i_t - \delta k_t \). Equation holds for each \( t \), and so we can write \( \Delta k = i - \delta k \).

- To preserve the current capital stock unchanged from this to the next period, i.e. to have \( \Delta k = 0 \), we need to have investment exactly equal to depreciation. So, we need to reach such a level of capital \( k^* \) that gives us \( sf(k^*) = i = \delta k^* \).

- Such a level of capital, \( k^* \) is labelled as the steady-state level of capital, and can be easily visualized at the graph.
Steady-state (SS) level of capital per worker $k^*$ is the one economy gravitates to in the long run regardless of its initial level of capital per worker be it above $k^*$, or below $k^*$.

At $k^*$, we can determine the steady-state (long-run) value of capital per worker, the long-run value of consumption per worker, and the long run value of investment per worker.

At steady state, output per worker and therefore standards of living stay the same over time.

With zero population and technological growth, the growth rate of total output at the steady state is zero.
Numerical Example

Let production function be $Y = F(K, L) = K^{1/2}L^{1/2}$. Is it the CRS? Check. Let $s = 0.3$, and $\delta = 0.1$. What does it mean?

In per capita terms, $Y/L = (K^{1/2}L^{1/2})/L$. And so $y = K^{1/2}L^{-1/2}$, or $y = (\frac{K}{L})^{1/2} = k^{1/2}$.

The law of motion of capital per worker is:
$\Delta k = sk^{1/2} - \delta k = 0.3k^{1/2} - 0.1k$.

Find the SS level of capital per worker, output per worker, and consumption per worker.

At the SS, $\Delta k = 0$. Thus, the SS $k^*$ solves:
$0.3(k^*)^{1/2} - 0.1k^* = 0$. And so $k^*/(k^*)^{1/2} = 0.3/0.1 = 3$. Thus, $k^* = 3^2 = 9$.

SS level of output per worker is just $(k^*)^{1/2} = 3$; SS level of consumption per worker is
$(1 - s)f(k^*) = 0.7(k^*)^{1/2} = 0.7 \times 3 = 2.1$
The Effect of Savings on Growth

- If savings rate increases from $s_1$ to $s_2$, we know that investment will be higher for any level of capital per worker, and this holds for the old SS $k^*$ as well.

- At the old SS $k^*$, investment will be higher than depreciation and so capital per worker will grow until it reaches the new SS.

- The path of the economy from the old SS to the new SS is called a **transitional path** of the economy.
The Effect of Savings on Growth

- Note that at the new SS, $i$ is higher and $y$ is higher. Yet the growth of $Y$ and $y$ are zero, as they are at the old SS. The growth rates are greater than zero during the transition from the old to the new SS!

- It is one of many rationales to encourage savings in the economy—to promote higher levels of output and standards of living.
Golden Rule of Capital

- We usually care about consumption which is a better measure of welfare than output.

- Imagine a social planner who wants to maximize consumption of the average worker, and who can set \( s \) and \( k \). What would he choose?

- Planner wants to maximize \( c = y - i = f(k) - sf(k) \). At the SS, \( c^* = f(k^*) - \delta k^* \) since \( sf(k^*) = \delta k^* \) at the SS.

- The maximum of \( c \) occurs at the level of \( k^*_\text{gold} \) where \( MPK(k^*_\text{gold}) = \delta \).

- More formally, \( k^*_\text{gold} \) solves \( \frac{\partial}{\partial k}[c] = 0 \), which happens at \( \frac{\partial}{\partial k}[f(k^*_\text{gold})] - \frac{\partial}{\partial k}[\delta k^*_\text{gold}] = 0 \), or at \( f'(k^*_\text{gold}) = \delta \).
Below the Golden Rule steady state, increases in steady-state capital raise steady-state consumption.

Above the Golden Rule steady state, increases in steady-state capital reduce steady-state consumption.
**Numerical Example**

To reach the golden rule level of capital per worker, the planner needs to induce the saving rate $s$ that will support this SS level of capital.

- Find $k_{gold}$ and $s_{gold}$ for the example above.
- At the SS: $s(k_{gold})^{1/2} = 0.1k_{gold}$. Thus, $s = 0.1 \times (k_{gold})^{1/2}$ (1).
- We also know that $MPK(k_{gold}) = \delta$.
- $MPK = f'(k)$. How to find $f'(k)$? For a power function, $f(x) = x^\alpha$, $f'(x) = \alpha x^{\alpha-1}$.
- Thus for our example $MPK = 1/2(k_{gold})^{-1/2}$, or $1/2 \times 1/(\sqrt{k_{gold}})$. And so... $\sqrt{k_{gold}} = 5$, and $k_{gold} = 25$.
- From (1), $s_{gold} = 0.1 \times 5 = 0.5$. 
Steady-state output, depreciation, and investment per worker

1. To reach the Golden Rule steady state...
2. ...the economy needs the right saving rate.
What if the economy has higher or lower savings rate than the one generating the golden rule level of capital and maximum consumption per worker?
The saving rate is increased.
Relaxing the assumption of no population growth

Let’s say that population in the economy grows $n$ per cent per year.

- We know that production and investment functions do not change, yet the ‘depreciation’ function changes.
- The capital per worker falls not only because the stock of capital wears off but also because, with positive population growth, there are more workers using the stock of capital.
- We can write the law of motion of capital as $\Delta k = i - (n + \delta)k$, or $\Delta k = sf(k) - (n + \delta)k$. 
Effects of Population Growth

If population growth changes from zero to some positive $n$:

- New steady-state $k^*$, and $y^*$ are lower.

- Golden rule level of $k$ is determined by maximizing $c = f(k_{gold}^*) - (n + \delta)k_{gold}^*$. WHY?

  It is determined by solving $MPK(k_{gold}^*) = n + \delta$.

- E.g., if $n=0.05$ in our example, new $k_{gold}^*$ is equal to 11.11. Check! Determine the golden rule savings rate.
1. An increase in the rate of population growth ...

2. ... reduces the steady-state capital stock.
Note that in the steady state of an economy with positive population growth:

1. $y$ does not grow. At odds with data since standards of living are changing over time.

2. $Y$ grows at the rate $n$. WHY?

Next, we will try to fix 1.

**Practice Problems:** 1a-c, 2, 3, 5–7.