Handout: PIH

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PIH

- 2-period problem, periods 0 and 1.
- Within-period utility function: $u(c_t) = -\frac{1}{2}(c_t - \bar{c})^2$.
- Freely borrow/lend at the constant real interest rate r.
- Endowments y_0 and y_1 are known at time 0.
- $\bar{c}$ is the "bliss" consumption level. If $c_t = \bar{c}$, a consumer attains the maximum utility possible, equal to 0.
- $\bar{c} \geq c_t$ so that the marginal utility is positive.
- $\beta \in (0, 1)$ is the time discount factor.
- $\beta(1 + r) = 1$.

Consumer wants to

$$\max_{c_0 \geq 0, c_1 \geq 0} U(c_0, c_1) = -\frac{1}{2}(c_0 - \bar{c})^2 - \beta\frac{1}{2}(c_1 - \bar{c})^2$$

$$\text{s.t. } c_0 + \frac{c_1}{1 + r} = y_0 + \frac{y_1}{1 + r}.$$

For this utility function, $MU_0 = \bar{c} - c_0$ and $MU_1 = \beta(\bar{c} - c_1)$. Thus, at the optimum (c_0^*, c_1^*), the following two equations should be satisfied:

$$(\overline{c} - c_0^*) = \beta(1 + r)(\overline{c} - c_1^*)$$
$$c_0^* + \frac{c_1^*}{1 + r} = y_0 + \frac{y_1}{1 + r}.$$

Since we assumed that $\beta = \frac{1}{1+r}$, we can write the first of those equations as $\overline{c} - c_0^* = \overline{c} - c_1^*$, or $c_0^* = c_1^*$. Plugging this equilibrium condition into the second equation, we obtain $c_0^* + \frac{c_0^*}{1+r} = y_0 + \frac{y_1}{1+r}$, or $c_0^* = c_1^* = \frac{1+r}{2+r}(y_0 + \frac{y_1}{1+r})$.

Consumer, *for these preferences*, will prefer to smooth consumption across periods perfectly.

Infinite horizon–1

Assume instead that a consumer's horizon is infinite, and s/he chooses consumption for periods $t = 0, 1, 2, \ldots$. In this case,

$$\begin{aligned}
\max_{c_0 \geq 0, c_1 \geq 0, c_2 \geq 0, \ldots} & U(c_0, c_1, c_2, \ldots) = -\frac{1}{2}(c_0 - \bar{c})^2 - \beta\frac{1}{2}(c_1 - \bar{c})^2 \\
& - \beta^2\frac{1}{2}(c_2 - \bar{c})^2 - \beta^3\frac{1}{2}(c_3 - \bar{c})^2 - \ldots \\
s.t.\ \ & c_0 + \frac{c_1}{1+r} + \frac{c_2}{(1+r)^2} + \frac{c_3}{(1+r)^3} + \ldots = y_0 + \frac{y_1}{1+r} + \frac{y_2}{(1+r)^2} + \\
& + \frac{y_3}{(1+r)^3} + \ldots
\end{aligned}$$

Infinite horizon–2

More compactly,

$$\max_{c_0 \geq 0, c_1 \geq 0, c_2 \geq 0, \ldots} U(c_0, c_1, c_2, \ldots) = \sum_{t=0}^{\infty} \left[-\frac{1}{2} \beta^t (c_t - \bar{c})^2 \right]$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} = \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t}.$$

Now, instead of finding just c_0^* and c_1^*, we will need to find the whole sequence $\{c_0^*, c_1^*, c_2^*, \ldots\}$.

Not so hard...Just need the (optimality) Euler equations and the lifetime budget constraint.

These equations should be satisfied at the optimum

$$\begin{aligned}
MU_1 &= (1+r)MU_2 \\
MU_2 &= (1+r)MU_3 \\
MU_3 &= (1+r)MU_4 \\
MU_4 &= (1+r)MU_5 \\
&\vdots \\
\sum_{t=0}^{\infty} \frac{c_t^*}{(1+r)^t} &= \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t}.
\end{aligned}$$

In terms of our utility function, the following equations should be satisfied at the optimum:

$$\begin{aligned}
\bar{c} - c_0^* &= \beta(1+r)(\bar{c} - c_1^*) \\
\beta(\bar{c} - c_1^*) &= \beta^2(1+r)(\bar{c} - c_2^*) \\
\beta^2(\bar{c} - c_2^*) &= \beta^3(1+r)(\bar{c} - c_3^*) \\
&\vdots \\
\sum_{t=0}^{\infty} \frac{c_t^*}{(1+r)^t} &= \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t}.
\end{aligned}$$

Since we assume that $\beta = \frac{1}{1+r}$:

$$c_0^* = c_1^*, c_1^* = c_2^*, c_2^* = c_3^* \dots$$

$$\sum_{t=0}^{\infty} \frac{c_t^*}{(1+r)^t} = \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t}.$$

Thus,

$$c^* \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} = \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t}.$$

Note that $\sum_{t=0}^{\infty} \frac{1}{(1+r)^t} = 1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \ldots,$ and $\frac{1}{1+r} < 1$. We want to find $S = 1 + x + x^2 + x^3 + \ldots,$ where $x \equiv \frac{1}{1+r}$. This sum will be equal to $\frac{1}{1-x} = \frac{1}{1-\frac{1}{1+r}} = \frac{1+r}{r}$.

$$c^* = c_0^* = c_1^* = c_2^* = \ldots = \underbrace{\frac{r}{1+r}\left[\sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t}\right]}_{y^p}.$$

Milton Friedman: individual consumption in each period should be related to an estimate of the permanent income.

Aside

It is easy to show that

$$S = 1 + x + x^2 + x^3 + x^4 + \ldots = \frac{1}{1-x}, \text{ for } \ |x| < 1.$$

Multiply the LHS and RHS of the equation by x and subtract the result from S, to obtain

$$S - xS = (1 + x + x^2 + x^3 + x^4 + \ldots) - (x + x^2 + x^3 + x^4 + \ldots) = 1.$$

Thus,

$$S = \frac{1}{1-x}.$$

Example

If $y_0 = y_1 = y_2 = \ldots = \overline{y}$, c^* will be equal to

$$\frac{r}{1+r}\overline{y}\left[1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \ldots\right] = \overline{y}\frac{r}{1+r}\frac{1+r}{r} = \overline{y}$$

In reality, future incomes are uncertain (that is, stochastic). At time t, we do not know for sure $\{y_{t+1}, y_{t+2}, y_{t+3}, \ldots\}$.

In this case, it does not make sense to set consumptions for periods c_{t+1}, $c_{t+2}, \ldots$ once and for all, since new information about future incomes and permanent income will arrive in periods following t.

In this case, the optimality (Euler) condition that links optimal consumptions in periods t and $t+1$, for the utility function we adopted, will read as:

$$c_t^* = E_t c_{t+1}^*.$$

Stochastic Euler equation

$$E_t \left(c^*_{t+1} - c^*_t \right) = E_t \Delta c^*_{t+1} = 0.$$

It means that the expected future change in consumption, given all the available information at time t, is equal to zero, that is consumption does not change between periods t and $t+1$ if there is no additional information arriving between periods t and $t+1$ about consumer's incomes. In statistics, a variable that has this property is called a *martingale*.

An implication of the martingale property of consumption is that consumption in period $t+1$ will differ from consumption in period t *only if* a consumer receives unexpected "news" about his permanent income.

In terms of the levels of consumption, we may derive the following relationship:

$$c_t = y_t^p = E_t \left[\frac{r}{1+r} \left(y_t + \frac{y_{t+1}}{1+r} + \frac{y_{t+2}}{(1+r)^2} + \frac{y_{t+3}}{(1+r)^3} + \dots \right) \right].$$

Important implications

Consumption will adjust by a larger margin if an unexpected change in income is permanent (e.g., disability vs. short spell of unemployment).

If the government contemplates about some policy affecting individual incomes (say, a tax cut) and wants to boost the economy via an increase in the aggregate consumption, it will only succeed if the policy affects permanent incomes a lot (say, a permanent reduction in income taxes). Otherwise, the reaction of consumers will be weak, if any.