ECON 581. Introduction to Arrow-Debreu Pricing and Complete Markets

Instructor: Dmytro Hryshko
Arrow-Debreu economy

- General equilibrium, exchange economy
- Static (all trades done at period 0) but multi-period
- No restrictions on preferences
Basic setting

- Two dates: 0, and 1. This set-up, however, is fully generalizable to multiple periods.
- $S$ possible states of nature at date 1, indexed by $s = 1, 2, \ldots, S$ with the corresponding probabilities $\pi(s)$.
- One perishable (=non storable) consumption good
- $I$ agents, indexed $i = 1, \ldots, I$, with preferences

$$u^i_0(c^i_0) + \beta^i \sum_{s=1}^{S} \pi(s)u^i(c^i_1(s))$$

- Agent $i$’s endowment is described by the vector $\{y^i_0, (y^i_1(s))_{s=1,2,\ldots,S}\}$
Traded securities

- **Arrow-Debreu securities (AD) (contingent claims):** security for state $s$ date 1 priced at time 0 at $q_1^0(s)$ promises delivery of one unit of commodity tomorrow (at date 1) if state $s$ is realized and nothing otherwise.

- Thus, individual $i$’s consumption in state $s$ will equal her holdings of AD securities for state $s$, date 1.
Agent’s problem. Competitive equilibrium setting

$$\max_{c^i_0, c^i_1(1), \ldots, c^i_S} u^i_0(c^i_0) + \beta^i \sum_{s=1}^{S} \pi(s) u^i(c^i_1(s))$$

s.t.

$$c^i_0 + \sum_{s=1}^{S} q^0_1(s) c^i_1(s) \leq y^i_0 + \sum_{s=1}^{S} q^0_1(s) y^i_1(s)$$

$$c^i_0, c^i_1(1), \ldots, c^i_S \geq 0$$
Definition of the equilibrium

Equilibrium is a set of contingent claim prices

\[ q^0_1(1), q^0_1(2), \ldots, q^0_1(S) \]

such that:

1. at those prices \( c^i_0, c^i_1(1), \ldots, c^i_1(S) \) solve problem (P) for all \( i \)'s, and

2. 

\[
\sum_{i=1}^{I} c^i_0 = \sum_{i=1}^{I} y^i_0, \quad \sum_{i=1}^{I} c^i_1(s) = \sum_{i=1}^{I} y^i_1(s), \quad \text{for each } s = 1, 2, \ldots, S.
\]
Competitive equilibrium and Pareto optimality illustrated

<table>
<thead>
<tr>
<th>Agents</th>
<th>Endowments</th>
<th>Preferences</th>
</tr>
</thead>
<tbody>
<tr>
<td>t=0</td>
<td>t=1</td>
<td></td>
</tr>
<tr>
<td>s=1</td>
<td>s=2</td>
<td></td>
</tr>
<tr>
<td>Agent 1</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>Agent 2</td>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>
Maximization problems

Agent 1:

\[
\max \left\{ c_0^1, c_1^1(1), c_1^1(2) \right\} \geq 0
\]
\[
\frac{1}{2} \left[ 10 + 1 \cdot q_1^0(1) + 2 \cdot q_1^0(2) - c_1^1(1) \cdot q_1^0(1) - c_1^1(2) \cdot q_1^0(2) \right]
\]
\[\underbrace{= c_0^1} \]
\[
+ 0.9 \left[ \frac{1}{3} \ln(c_1^1(1)) + \frac{2}{3} \ln(c_1^1(2)) \right]
\]

Agent 2:

\[
\max \left\{ c_0^2, c_1^2(1), c_1^2(2) \right\} \geq 0
\]
\[
\frac{1}{2} \left[ 5 + 4 \cdot q_1^0(1) + 6 \cdot q_1^0(2) - c_1^2(1) \cdot q_1^0(1) - c_1^2(2) \cdot q_1^0(2) \right]
\]
\[\underbrace{= c_0^2} \]
\[
+ 0.9 \left[ \frac{1}{3} \ln(c_1^2(1)) + \frac{2}{3} \ln(c_1^2(2)) \right]
\]
Optimum

- **Optimality conditions:**

  **Agent 1:** \[
  \begin{align*}
  c_1^1(1) & : \quad q_1^0(1) = 0.9 \cdot \frac{1}{3} \cdot \frac{1}{c_1^1(1)} \\
  c_1^1(2) & : \quad q_1^0(2) = 0.9 \cdot \frac{2}{3} \cdot \frac{1}{c_1^1(2)} \\
  c_1^2(1) & : \quad q_1^0(1) = 0.9 \cdot \frac{1}{3} \cdot \frac{1}{c_1^2(1)} \\
  c_1^2(2) & : \quad q_1^0(2) = 0.9 \cdot \frac{2}{3} \cdot \frac{1}{c_1^2(2)}
  \end{align*}
  \]

  **Agent 2:** \[
  \begin{align*}
  c_1^2(1) & : \quad q_1^0(1) = 0.9 \cdot \frac{1}{3} \cdot \frac{1}{c_1^2(1)} \\
  c_1^2(2) & : \quad q_1^0(2) = 0.9 \cdot \frac{2}{3} \cdot \frac{1}{c_1^2(2)}
  \end{align*}
  \]

- **Feasibility conditions:**

  \[
  \begin{align*}
  c_1^1(1) + c_1^2(1) &= 5 \\
  c_1^1(2) + c_1^2(2) &= 8 \\
  c_1^1(1) &= c_1^2(1) = 2.5 \\
  c_1^1(2) &= c_1^2(2) = 4.
  \end{align*}
  \]
Prices of AD securities

Optimality conditions can be expressed as

\[
q_1^0(s) = \frac{0.9 \cdot \pi(s) \cdot \frac{1}{c_1^i(s)}}{1/2}, \quad s, i = 1, 2, \text{ or}
\]

\[
q_1^0(s) = \frac{\beta \cdot \pi(s) \cdot \frac{\partial u^i}{\partial c_1^i(s)}}{\frac{\partial u^i_0}{\partial c_0^i}}, \quad s, i = 1, 2.
\]

That is,

\[
\frac{\text{today's price of the tomorrow's good if state } s \text{ is realized}}{\text{price of the today's good}} = \frac{\text{MU}_1^i(s)}{\text{MU}_0^i}
\]

\[
q_1^0(1) = 2 \cdot 0.9 \cdot \frac{1}{3} \cdot \frac{1}{c_1^1(1)} = 2 \cdot 0.9 \cdot \frac{1}{3} \cdot \frac{1}{2.5} = 0.24
\]

\[
q_1^0(2) = 2 \cdot 0.9 \cdot \frac{2}{3} \cdot \frac{1}{c_1^1(2)} = 2 \cdot 0.9 \cdot \frac{2}{3} \cdot \frac{1}{4} = 0.30
\]
Notes on AD prices

- Prices reflect probabilities, and marginal rates of substitution and therefore relative scarcities of the goods (total quantities of goods differ in different states)
- If date 1 marginal utilities were constant (linear, risk neutral preferences), the goods endowments wouldn’t influence the AD prices, which would be then exactly proportional to the state probabilities
Post-trade equilibrium consumptions

<table>
<thead>
<tr>
<th></th>
<th>t=0</th>
<th>t=1</th>
<th>utility</th>
<th></th>
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<tbody>
<tr>
<td>s=1</td>
<td></td>
<td></td>
<td></td>
<td>s=2</td>
</tr>
<tr>
<td>Agent 1</td>
<td>9.04</td>
<td>2.5</td>
<td>4</td>
<td>5.62</td>
</tr>
<tr>
<td>Agent 2</td>
<td>5.96</td>
<td>2.5</td>
<td>4</td>
<td>4.09</td>
</tr>
<tr>
<td>Total</td>
<td>15</td>
<td>5</td>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>

\[
c_0^1 = 10 + 1 \cdot 0.24 + 2 \cdot 0.3 - 2.5 \cdot 0.24 - 4 \cdot 0.3 = 9.04
\]

\[
c_0^2 = 5 + 4 \cdot 0.24 + 6 \cdot 0.3 - 2.5 \cdot 0.24 - 4 \cdot 0.3 = 5.96
\]
Is the equilibrium Pareto optimal? (If yes, it is impossible to rearrange the allocation of consumptions so that the utility of one agent is higher without reducing the utility of the other agent.)
Pareto problem

\[
\begin{align*}
\max_{\{c_0^1, c_1^1(1), c_1^1(2)\} \geq 0} & \quad u^1(c_0^1, c_1^1(1), c_1^1(2)) + \lambda u^2(c_0^2, c_1^2(1), c_1^2(2)) \\
\text{s.t.} & \quad c_0^1 + c_0^2 = 15 \quad c_1^1(1) + c_1^2(1) = 5 \quad c_1^1(2) + c_1^2(2) = 8 \\
& \quad c_0^1, c_1^1(1), c_1^1(2), c_0^2, c_1^2(1), c_1^2(2) \geq 0
\end{align*}
\]

FOCs:

\[
\frac{u_0^1}{u_0^2} = \frac{u_1^1(1)}{u_1^2(1)} = \frac{u_1^1(2)}{u_1^2(2)} = \lambda.
\]
In terms of our example, the first 3 equalities are

\[
\frac{1/2}{1/2} = \frac{0.9 \cdot \frac{1}{3} \cdot \frac{1}{c_1^1(1)}}{0.9 \cdot \frac{1}{3} \cdot \frac{1}{c_1^2(1)}} = \frac{0.9 \cdot \frac{2}{3} \cdot \frac{1}{c_1^1(2)}}{0.9 \cdot \frac{2}{3} \cdot \frac{1}{c_1^2(2)}}
\]

In our example, competitive equilibrium corresponds to the Pareto optimum with equal weighting of the two agents’ utilities, \( \lambda = 1 \).
Incomplete markets=less AD securities than states
Assume that only state-1 date-1 AD security is available.

Agent 1:

\[
\max_{\{c_0^1, c_1^1(1)\} \geq 0} \frac{1}{2} \left[ 10 + 1 \cdot q_1^0(1) - c_1^1(1) \cdot q_1^0(1) \right] = c_0^1
\]

\[
+ 0.9 \left[ \frac{1}{3} \ln(c_1^1(1)) + \frac{2}{3} \ln(2) \right] = \beta^1
\]

Agent 2:

\[
\max_{\{c_0^2, c_1^2(1)\} \geq 0} \frac{1}{2} \left[ 5 + 4 \cdot q_1^0(1) - c_1^2(1) \cdot q_1^0(1) \right] = c_0^2
\]

\[
+ 0.9 \left[ \frac{1}{3} \ln(c_1^2(1)) + \frac{2}{3} \ln(6) \right] = \beta^2
\]
Post-trade allocation

FOCs and the feasibility condition imply
\[ c_1^1(1) = c_1^2(1) = (1 + 4)/2 = 2.5. \]

<table>
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<td></td>
<td>s=1</td>
<td>s=2</td>
<td></td>
</tr>
<tr>
<td>Agent 1</td>
<td>9.64</td>
<td>2.5</td>
<td>2</td>
</tr>
<tr>
<td>Agent 2</td>
<td>5.36</td>
<td>2.5</td>
<td>6</td>
</tr>
<tr>
<td>Total</td>
<td>15</td>
<td>5</td>
<td>8</td>
</tr>
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The market with AD securities for each state, called complete market, is Pareto superior to the incomplete market.
Risk sharing. New endowment matrix

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<td>t=0</td>
<td>t=1</td>
</tr>
<tr>
<td>Agent 1</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Agent 2</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>Agents</td>
<td>Utilities</td>
<td>Expected utility in period 1</td>
</tr>
<tr>
<td>--------</td>
<td>------------</td>
<td>-------------------------------</td>
</tr>
<tr>
<td></td>
<td>$s=1$</td>
<td></td>
</tr>
<tr>
<td>Agent 1</td>
<td>$\ln(1)$</td>
<td>$\frac{1}{2} \ln(1) + \frac{1}{2} \ln(5) = 0.8047$</td>
</tr>
<tr>
<td></td>
<td>$s=2$</td>
<td></td>
</tr>
<tr>
<td>Agent 2</td>
<td>$\ln(5)$</td>
<td>$\frac{1}{2} \ln(5) + \frac{1}{2} \ln(1) = 0.8047$</td>
</tr>
</tbody>
</table>

Table 1: No trade
**Table 2: Trade under complete markets**

<table>
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<tr>
<th>Agents</th>
<th>Utilities</th>
<th>Expected utility in period 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>t=1</td>
<td>s=1</td>
<td>$\ln(3)$</td>
</tr>
<tr>
<td>s=2</td>
<td></td>
<td>$\ln(3)$</td>
</tr>
</tbody>
</table>

Agent 1: $\ln(3) + \frac{1}{2} \ln(3) + \frac{1}{2} \ln(3) = 1.099$

Agent 2: $\ln(3) + \frac{1}{2} \ln(3) + \frac{1}{2} \ln(3) = 1.099$

- Both agents are perfectly insured=no variation in tomorrow’s consumption regardless of the realized state of nature.
- This happens because the aggregate endowment in state 1 and 2 is the same (=6), that is there’s no aggregate risk.
Notes on Pareto optimal allocations

\[ \lambda = \frac{u_0^1}{u_0^2} = \frac{u_1^1(1)}{u_1^2(1)} = \frac{u_1^1(2)}{u_1^2(2)} \iff \frac{u_1^1(1)}{u_1^1(2)} = \frac{u_2^1(1)}{u_2^1(2)} \]

- If one of the two agents is fully insured—no variation in her date 1 consumption (MRS=1)—the other must be as well.
Notes on Pareto optimal allocations

\[ \lambda = \frac{u_0^1}{u_0^2} = \frac{u_1^1(1)}{u_1^2(1)} = \frac{u_1^1(2)}{u_1^2(2)} \iff \frac{u_1^1(1)}{u_1^2(1)} = \frac{u_1^2(1)}{u_1^2(2)} \]

- If one of the two agents is fully insured—no variation in her date 1 consumption (MRS=1)—the other must be as well.
- More generally, if the MRS are to differ from 1, given that they must be equal between the agents, the low consumption-high MU state must be the same for both agents and similarly for the high consumption-low MU state. Impossible when there’s no aggregate risk, hence individuals are perfectly insured in the absence of aggregate risk.

If there is aggregate risk, however, the above reasoning also implies that, at a Pareto optimum, it is shared “proportionately” among agents with same risk tolerance.

If agents are differentially risk averse, in a Pareto optimal allocation the less risk averse will typically provide some insurance services to the more risk averse.

More generally, optimal risk sharing dictates that the agent most tolerant of risk bears a disproportionate share of it.
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Notes on Pareto optimal allocations

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- If agents are differentially risk averse, in a Pareto optimal allocation the less risk averse will typically provide some insurance services to the more risk averse.
- More generally, optimal risk sharing dictates that the agent most tolerant of risk bears a disproportionate share of it.
CRRA preferences

Let $u^i_0$ and $u^i(c^i_1(s))$ be CRRA, and assume homogeneous time discounting factors:

\[ u(c) = \frac{c^{1-\rho}}{1 - \rho}, \quad \rho > 0, \quad \rho \neq 1 \]

\[ u(c) = \log c, \quad \rho = 1. \]
AD prices with homogeneous CRRA preferences

Recall FOCs:

\[ q_1^0(s)(c_0^i)^{-\rho} = \pi(s)\beta(c_1^i(s))^{-\rho}, \quad s = 1, \ldots, S, \quad i = 1, \ldots, I \]

\[ \Rightarrow c_1^i(s) = \left[ \frac{\pi(s)\beta}{q_1^0(s)} \right]^{1/\rho} c_0^i, \quad s = 1, \ldots, S, \quad i = 1, \ldots, I \]

\[ \Rightarrow \sum_i c_1^i(s) = \left[ \frac{\pi(s)\beta}{q_1^0(s)} \right]^{1/\rho} \sum_i c_0^i, \quad s = 1, \ldots, S, \quad i = 1, \ldots, I \]

\[ \Rightarrow \bar{y}_1(s) = \bar{y}_0 \]

\[ \Rightarrow q_1^0(s) = \pi(s)\beta \left[ \frac{\bar{y}_1(s)}{\bar{y}_0} \right]^{-\rho}, \quad s = 1, \ldots, S, \]

where \( \bar{y}_0 \) and \( \bar{y}_1(s) \) are total, economy-wide, endowments at date 0, and date 1, state \( s \), respectively.
The existence of the representative consumer. CRRA preferences

\[ q_1^0(s) = \pi(s)\beta \left[ \frac{\bar{y}_1(s)}{\bar{y}_0} \right]^{-\rho}, \quad s = 1, \ldots, S \]

Notice that the economy with the representative consumer who owns the economy-wide endowments at each date-state will result into the same equilibrium vector of prices and aggregate consumption as a decentralized economy populated by consumers with

- identical time discount factors and
- identical CRRA preferences.
Equilibrium consumption levels: CRRA preferences

\[
\frac{\pi(s)\beta(c^i_1(s))^{-\rho}}{(c^i_0)^{-\rho}} = q^0_1(s) = \frac{\pi(s)\beta(c^j_1(s))^{-\rho}}{(c^j_0)^{-\rho}}
\]

\[
\Rightarrow \frac{c^i_1(s)}{c^i_0} = \frac{\bar{y}_1(s)}{\bar{y}_0} = \frac{c^j_1(s)}{c^j_0}
\]

\[
\Rightarrow \frac{c^i_1(s)}{\bar{y}_1(s)} = \frac{c^i_0}{\bar{y}_0} \quad \text{and} \quad \frac{c^j_1(s)}{\bar{y}_1(s)} = \frac{c^j_0}{\bar{y}_0}
\]

Furthermore,

\[
\frac{\pi(s)\beta(c^i_1(s))^{-\rho}}{\pi(s')\beta(c^i_1(s'))^{-\rho}} = \frac{q^0_1(s)}{q^0_1(s')} = \frac{\pi(s)\beta(c^j_1(s))^{-\rho}}{\pi(s')\beta(c^j_1(s'))^{-\rho}}
\]

\[
\Rightarrow \frac{c^i_1(s)}{c^i_1(s')} = \frac{q^0_1(s)}{q^0_1(s')} = \frac{\bar{y}_1(s)}{\bar{y}_1(s')} = \frac{c^j_1(s)}{c^j_1(s')}
\]

\[
\Rightarrow \frac{c^i_1(s)}{\bar{y}_1(s)} = \frac{c^i_1(s')}{\bar{y}_1(s')} \quad \text{and} \quad \frac{c^j_1(s)}{\bar{y}_1(s)} = \frac{c^j_1(s')}{\bar{y}_1(s')}
\]
Summary

- Any agent $i$’s consumption is a constant share $\kappa_i$ of date 1 total endowment regardless of the state.
- Any agent $i$’s date 0 consumption share in total output is the same as her date 1 share.
- Agent $i$’s share in aggregate consumption/wealth is the agent’s share of the aggregate wealth on date 0, evaluated at equilibrium Arrow-Debreu prices.
Individual consumption share: CRRA preferences

\[ y_0^i + \sum_{s=1}^{S} q_1^0(s)y_1^i(s) = c_0^i + \sum_{s=1}^{S} q_1^0(s)c_1^i(s) \]

\[ = \kappa_i \bar{y}_0 + \sum_{s=1}^{S} q_1^0(s)\kappa_i \bar{y}_1(s) \]

\[ = \kappa_i \left[ \bar{y}_0 + \sum_{s=1}^{S} q_1^0(s)\bar{y}_1(s) \right] \]

\[ \kappa_i = \frac{y_0^i + \sum_{s=1}^{S} q_1^0(s)y_1^i(s)}{\bar{y}_0 + \sum_{s=1}^{S} q_1^0(s)\bar{y}_1(s)} \]

You can further simplify the above expression by plugging in the AD prices.
Suggested readings for this lecture