A Note on the Solow Model.

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Households and production. We assume that a closed economy produces one unique final good. A representative firm utilizes a production function $F(K, L)$ to produce all of the economy's output, Y. We assume that production function is constant returns to scale in K and L: $F(zK, zL) = zF(K, L) = zY$.[1] That is, doubling ($z = 2$) the use of K and L doubles the amount of output produced.

Market structure. We assume that households are homogeneous (identical) and save a constant fraction of income, s. Households own labor and capital in the economy. At time t, the representative firm purchases one unit of L at the labor market at the price $w(t)$. The real rental price of one unit of capital at time t is equal to $R(t) = r(t) + \delta$, where $r(t)$ is the real interest rate and δ is the depreciation rate (see our notes on investment). Households purchase the final good for consumption and investment purposes. We set the price of the final good equal to one in all periods. Thus, $w(t)$ is the real wage and it is measured in terms of the amount of final goods afforded per one hour worked. Similarly, $R(t)$ is the amount of final goods per one unit of capital rented out to the firm. We assume that labor and capital markets, and the final goods market are competitive.

At each point in time, a competitive firm maximizes profits

$$\max_{K>0, L>0} \pi = Y - wL - RK = \underbrace{F(K, L)}_{\text{revenue}} - \underbrace{(wL + RK)}_{\text{costs}}. \tag{1}$$

The firm's objective is to choose non-negative amounts of K and L that bring the maximum profit. Denote the profit-maximizing choice as (K^*, L^*); $F_K(K, L)$ as the marginal product of capital, and $F_L(K, L)$ as the marginal product of labor. At this choice, the following two conditions should be simultaneously satisfied:

$$F_K(K^*, L^*) = R \tag{2}$$

$$F_L(K^*, L^*) = w. \tag{3}$$

[1] Note that production function is increasing returns to scale in K and L if $F(zK, zL) > zY$; decreasing returns to scale—if $F(zK, zL) < zY$.

Consider the first equation. If it is not satisfied, say $F_K(K^*, L^*) > R$, then the marginal product of capital is higher than the marginal cost of capital and therefore raising K^* marginally will bring more profits, which contradicts our assumption that K^* is optimal.

Thus, in the equilibrium, $\pi = F(K,L) - F_K(K,L)K - F_L(K,L)L$. Notice that we ignored stars in notation to highlight the fact that the observed choices of K and L are always optimal. The assumption of perfect competition and constant returns to scale production function imply zero profits for the representative firm. This result follows from the Euler theorem. Thus, factor payments exhaust the total revenue, and, at each point in time, $Y = wL + RK$.

Capital is accumulated in the economy in accordance with the following equation:

$$\Delta K = I - \delta K, \tag{4}$$

where I is the aggregate investment in the economy and δ is the depreciation rate. We assume that the economy is endowed with some positive amount of capital and labor at its initiation. Since the economy is closed all household's savings are channeled into domestic investment and so

$$\Delta K = sY - \delta K = sF(K,L) - \delta K, \tag{5}$$

where, as noted before, s is the constant savings rate.

Equilibrium. Equilibrium in the economy is defined as the path of *allocations*, C, Y, K, and *prices* w and R, given the amount of labor resources and the initial capital stock.

Define output per worker as $y_{pw} = \frac{Y}{L}$, and capital per worker as $k_{pw} = \frac{K}{L}$. By our assumption on the production function, $zY = F(zK, zL)$. Let $z \equiv \frac{1}{L}$. Thus, $y_{pw} = \frac{Y}{L} = F(\frac{K}{L}, \frac{L}{L}) = f(k_{pw})$, where we assumed that $F(\frac{K}{L}, 1) = f(k_{pw})$. From our calculus note, we know that $\frac{\Delta k_{pw}}{k_{pw}} = \frac{\Delta K}{K} - \frac{\Delta L}{L} = \frac{sF(K,L)-\delta K}{K} - \frac{\Delta L}{L} = \frac{sF(K,L)/L}{K/L} - \delta - \frac{\Delta L}{L} = \frac{sy_{pw}}{k_{pw}} - (\delta + n)$. We have made an assumption that working population is growing at a constant rate n. Thus, $\Delta k_{pw} = sy_{pw} - \delta k_{pw} = sf(k_{pw}) - (\delta + n)k_{pw}$. This is the law of motion for capital per worker in the economy without technological growth. We argued in class that the economy will tend to the steady state equilibrium where $\Delta k_{pw} = 0$, or when $sf(k^*_{pw}) = (n + \delta)k^*_{pw}$.

The term $(n + \delta)k_{pw}$ can be called the *break-even investment*. It measures the amount of investment per worker needed to maintain capital per worker at its previous level k_{pw}.

Assume, for example, that $k_{pw} = 10$, $\delta = 0.1$, and $n = 1$. Using capital implies that in the next period, for a fixed population size, capital per worker will fall by $\delta k_{pw} = 0.1 * 10 = 1$ unit. Furthermore, population will double next period, and, to preserve the same capital per worker in the next period, we will need to "endow" each new worker with exactly the same $k_{pw} = 10$. Thus, the total investment we need to undertake so that $\Delta k_{pw} = 0$, that is for k_{pw} to stay constant, is equal to $(n+\delta)k_{pw} = 11$ units.

Example. Let production function be of Cobb-Douglas type $Y = K^{\alpha}L^{1-\alpha}$. It follows that $y_{pw} = f(k_{pw}) = (k_{pw})^{\alpha}$. The steady-state equilibrium in this economy is defined from $s(k^*_{pw})^{\alpha} = (n+\delta)k^*_{pw}$. Thus, $k^*_{pw} = \left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}}$. Also, $y^*_{pw} = (k^*_{pw})^{\alpha} = \left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$, $c^*_{pw} = (1-s)\left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$. Note that income per worker in the steady state will be higher if s is higher, n is lower, δ is lower, and α is higher. Thus, countries with higher investment rates, lower population growth rates, lower depreciation rates, and lower degree of the decline in the marginal product of capital will be richer relative to the otherwise similar countries.

We found steady-state *allocations*, what are the equilibrium prices, w and R? We know that $w = F_L(K^*, L^*) = (1-\alpha)K^{\alpha}L^{-\alpha} = (1-\alpha)\left(\frac{K}{L}\right)^{\alpha} = (1-\alpha)(k_{pw})^{\alpha} = (1-\alpha)y_{pw} = (1-\alpha)\frac{Y}{L}$. The rental price of capital $R = F_K(K^*, L^*) = \alpha K^{\alpha-1}L^{1-\alpha} = \alpha\left(\frac{K}{L}\right)^{\alpha-1} = \alpha\frac{y_{pw}}{k_{pw}} = \alpha\frac{Y/L}{K/L} = \alpha\frac{Y}{K}$. The real interest rate is equal to $r = F_K(K, L) - \delta = \alpha\frac{y_{pw}}{k_{pw}} - \delta$.

Thus, the equilibrium rental prices, for a Cobb-Douglas production function, are linked to the average productivity of each factor of production. In the steady-state equilibrium, $w^* = (1-\alpha)y^*_{pw}$ and $R^* = \alpha\frac{y^*_{pw}}{k^*_{pw}}$.

We can write our equilibrium allocations and prices in the steady state as functions of exogenous parameters only: $k^*_{pw} = k^*_{pw}(s, n, \delta)$, $y^*_{pw} = y^*_{pw}(s, n, \delta)$, $R^* = R^*(s, n, \delta)$, $w^* = w^*(s, n, \delta)$.

For our example, $w^* = (1-\alpha)\left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$, $R^* = \alpha\frac{\left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}}{\left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}}} = \alpha\left(\frac{s}{n+\delta}\right)^{-1} = \alpha\frac{n+\delta}{s}$, and $r^* = \frac{n+\delta}{s} - \delta$. Thus, in the steady state of the economy without technological progress, real wages and real interest rates are constant: they are just functions of exogenous parameters of the model, s, δ, n and α, assumed to be constant. Note that, ceteris paribus, the real interest rate is higher for lower saving rates, higher population growth rates, and higher depreciation rates. Intuitively, more people in the economy (i.e., higher n) will make a marginal unit of capital more productive.[2]

[2] Think of intuitions for why the real interest rate is higher, ceteris paribus, if the depreciation rate is higher, and savings rate is lower.

What is the share of capital and labor costs in total income? We know that $Y = wL + RK$, and the share of labor costs/income and capital costs/income in total income are equal to $\frac{wL}{Y}$ and $\frac{RK}{Y}$, respectively. Thus, $1 = \frac{wL}{Y} + \frac{RK}{Y} = \frac{wL/L}{Y/L} + \frac{RK/L}{Y/L} = \frac{w}{y_{pw}} + \frac{Rk_{pw}}{y_{pw}} = \frac{(1-\alpha)y_{pw}}{y_{pw}} + \alpha \frac{y_{pw}}{k_{pw}} \frac{k_{pw}}{y_{pw}} = (1-\alpha) + \alpha$. Thus, for a Cobb-Douglas production function $Y = K^\alpha L^{1-\alpha}$, the share of labor and capital costs in total income are constant, and equal to $(1-\alpha)$ and α, respectively. For developed economies, we see that these shares, although fluctuating over time, tend to 2/3 and 1/3. Hence, if we think of the aggregate production function in developed economies being of Cobb-Douglas type, we should assume that $\alpha = 1/3$.

Golden-rule savings rate. It can be shown that if production function is $Y = K^\alpha L^{1-\alpha}$, then the golden-rule savings rate in the economy is equal to α—the share of capital income in total income. In other words, if the economy saves at the rate $s = \alpha$, then it will reach the golden-rule steady state. For the economy without technological progress, the golden-rule capital per worker is obtained from $MPK(k^*_{gold}) = n + \delta$. For this production function, $MPK = F_K(K, L) = \alpha K^{\alpha-1} L^{1-\alpha} = \alpha \left(\frac{K}{L}\right)^{\alpha-1} = \alpha k_{pw}^{\alpha-1}$. Thus, the golden rule capital per worker is equal to $\left(\frac{\alpha}{n+\delta}\right)^{\frac{1}{1-\alpha}}$. If the economy saves its capital income (and, correspondingly, consumes all of its labor income), the total savings in the economy are αY, the per worker savings are αy_{pw}. For this economy, the steady state happens when $\alpha (k^*_{pw})^\alpha = (n+\delta) k^*_{pw}$, i.e., when $k^*_{pw} = \left(\frac{\alpha}{n+\delta}\right)^{\frac{1}{1-\alpha}}$, which is exactly equal to the golden rule capital per worker we've just found.[3]

Transitional dynamics. Assume that the economy is in the steady state defined by k^*_{pw}. What will happen to the economy if we change one of the parameters, say, savings rate from s to s', with $s' > s$? We know that the economy will move towards $k^{**}_{pw} > k^*_{pw}$ and $y^{**}_{pw} > y^*_{pw}$. We are now interested in the transitional path of the economy from k^*_{pw} to k^{**}_{pw}. Note that at the old steady state, $sy^*_{pw} - (n+\delta)k^*_{pw} = 0$. Since we are increasing savings rate, we are redistributing a higher share of income per capita towards investment, reducing consumption at the same time. Thus, at the instant the economy raises its savings rate, investment per worker becomes $s'y^*_{pw}$; k^*_{pw} does not change at the instant since it takes one period for investment to enlarge the capital stock. Thus, $s'y^*_{pw} - (n+\delta)k^*_{pw} > sy^*_{pw} - (n+\delta)k^*_{pw}$ since $s' > s$, by assumption. The percentage change of capital per worker at the time of the change is equal to $\frac{\Delta k_{pw}}{k^*_{pw}} = \frac{s'y^*_{pw}}{k^*_{pw}} - (n+\delta) = s'\frac{n+\delta}{s} - (n+\delta) > 0$ since $s' > s$. Since Δk_{pw} will be positive until the economy hits k^{**}_{pw}, the growth of capital per worker will be positive and shrinking towards 0, with the largest growth rate observed at the time of the change in the savings rate. Since total capital in the economy is $K = k_{pw}L$, the growth rate in total capital is $\frac{\Delta K}{K} = \frac{\Delta k_{pw}}{k_{pw}} + \frac{\Delta L}{L}$. In the old steady state, $\frac{\Delta K}{K} = n$, at the time of the change in the savings rate, $\frac{\Delta K}{K} = s'\frac{n+\delta}{s} - (n+\delta) + n = s'\frac{n+\delta}{s} - \delta$. Since $y_{pw} = k^\alpha_{pw}$, the growth rate in output per

[3]The same finding will hold for $Y = K^\alpha (EL)^{1-\alpha}$. See below for details on this production function.

worker is equal to $\alpha\frac{\Delta k_{pw}}{k_{pw}}$. At the time of the change, $\frac{\Delta y_{pw}}{y_{pw}} = \alpha s' \frac{n+\delta}{s} - \alpha(n+\delta)$. The economy's output per worker grows, first, at a zero rate before the change, at the rate $\alpha s' \frac{n+\delta}{s} - \alpha(n+\delta)$ at the time of the change, and steadily shrinks towards zero, when the economy is at k_{pw}^{**}. Since total output is $Y = y_{pw}L$, it grows at the rate $\frac{\Delta y_{pw}}{y_{pw}} + n$.

Analogously, we can trace out the effects of exogenous changes in other parameters, n, g, or δ. For example, assume that the economy is in the steady state and population growth increases to $n' > n$, while the other parameters are kept constant. At the time of the change, capital per worker, and output per worker (and therefore investment and savings per worker) will stay unchanged. This follows from the fact that it takes one time period (here it is better to think in terms of one instant rather than one year as the period of economy's observation) for the economy to enlarge its labor force, now at a faster rate than before. Thus, at the time of the change $\Delta k_{pw} = sy_{pw}^* - (n' + \delta)k_{pw}^*$, and we know that $sy_{pw}^* = (n+\delta)k_{pw}^*$ since the economy was in the steady state before the arrival of "news" about its population growth. Thus, at the time of the change, $\Delta k_{pw} = (n+\delta)k_{pw}^* - (n'+\delta)k_{pw}^* = (n-n')k_{pw}^*$ and the percentage change in capital per worker is $\frac{\Delta k_{pw}}{k_{pw}^*} = n - n' < 0$, since we assumed that $n' > n$. We also know that k_{pw}^{**}—the new steady state of the economy—will be lower than k_{pw}^*. During the economy's transition, the growth rate in capital per worker will be $n - n'$ at the time of the change and will start increasing and eventually approach zero, when the economy hits its new steady state k_{pw}^{**}.

Solow model with technological growth.

We assume that technological progress is labor-augmenting so that $Y = F(K, EL)$, where E is the "level of technology" at a particular point in time, and production function is constant returns to scale in K and L. This means that $zY = F(zK, EzL)$.

If we set $z = \frac{1}{EL}$, then $\frac{Y}{EL} = F(\frac{K}{EL}, \frac{EL}{EL}) = F(\frac{K}{EL}, 1) = f(\frac{K}{EL})$. Denote the per effective (efficient) worker output as $y_{pew} = \frac{Y}{EL}$ and per effective worker capital as $k_{pew} = \frac{K}{EL}$. Then $y_{pew} = f(k_{pew})$. If, for example, $Y = K^{\alpha}(EL)^{1-\alpha}$, then we can show that $zY = (zK)^{\alpha}(EzL)^{1-\alpha}$. Setting z to $\frac{1}{EL}$, we obtain $\frac{Y}{EL} = y_{pew} = (\frac{K}{EL})^{\alpha}(\frac{EL}{EL})^{1-\alpha} = k_{pew}^{\alpha}$. Output per effective worker is a function of capital per effective worker only. We assume that technology grows at an exogenous rate g, i.e., $\frac{\Delta E}{E} = g$.

From our calculus note, $\frac{\Delta k_{pew}}{k_{pew}} = \frac{\Delta K}{K} - \frac{\Delta L}{L} - \frac{\Delta E}{E} = s\frac{F(K,EL)}{K} - (n+g+\delta) = s\frac{F(K,EL)/EL}{K/EL} - (n+g+\delta) = s\frac{f(k_{pew})}{k_{pew}} - (n+g+\delta)$. Thus, the law of motion of capital per effective worker is $\Delta k_{pew} = sf(k_{pew}) - (n+g+\delta)k_{pew}$. Similarly to our previous analysis, the economy will reach the steady state when $\Delta k_{pew} = 0$, and so $sf(k_{pew}^*) = (n+g+\delta)k_{pew}^*$. For our production function $Y = K^\alpha(EL)^{1-\alpha}$, $y_{pew} = k_{pew}^\alpha$. Thus, $k_{pew}^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$ and $y_{pew}^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}$—the functions of exogenous parameters only. Note that capital per worker $k_{pw} = \frac{K}{L} = k_{pew}E$, and $y_{pw} = \frac{Y}{L} = y_{pew}E$. Thus, $\frac{\Delta k_{pw}}{k_{pw}} = \frac{\Delta k_{pew}}{k_{pew}} + \frac{\Delta E}{E} = \frac{\Delta k_{pew}}{k_{pew}} + g$, and $\frac{\Delta y_{pw}}{y_{pw}} = \frac{\Delta y_{pew}}{y_{pew}} + g$. In the steady state, $\frac{\Delta k_{pew}}{k_{pew}} = \frac{\Delta y_{pew}}{y_{pew}} = 0$, and so output per worker and capital per worker both *grow* at the rate g. Aggregate capital is $K = k_{pew}EL$ and aggregate output is $Y = y_{pew}EL$. Thus, the growth rates in total output and total capital are, respectively, equal to $\frac{\Delta Y}{Y} = \frac{\Delta y_{pew}}{y_{pew}} + g + n$, $\frac{\Delta K}{K} = \frac{\Delta k_{pew}}{k_{pew}} + g + n$. In the steady state, both of them will equal to $g+n$, since neither capital per effective worker nor output per effective worker are growing in the steady state (sometimes called a *balanced growth path* since aggregates relative to labor are actually growing).

Set the time when the economy starts operating to 0. Then $L(1) = (1+n)L(0)$, $L(2) = (1+n)L(1) = (1+n)(1+n)L(0) = (1+n)^2L(0)$, $\ldots$, $L(t) = (1+n)^tL(0)$, where $L(t)$ is the amount of labor resources at time t, in t years after the economy started operating. Similarly, $E(t) = (1+g)^tE(0)$. Thus, *capital per worker*, in t years from the economy's initiation, is $\left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} E(0)(1+g)^t$, and *total capital* at t is $\left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} E(0)L(0)(1+g)^t(1+n)^t$. *Output per worker* at time t will be $\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} E(0)(1+g)^t$ and *total output* will be $\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} E(0)L(0)(1+n)^t(1+g)^t$.

What are the equilibrium prices in the economy with technological growth? Recall that $w = F_L(K, EL)$ and $R = F_K(K, EL)$. For our production function, $F_L(K, EL) = (1-\alpha)K^\alpha E^{1-\alpha}L^{-\alpha} = (1-\alpha)\frac{K^\alpha(EL)^{1-\alpha}}{L} = (1-\alpha)\frac{Y}{L} = (1-\alpha)y_{pw}$. Thus, $w = (1-\alpha)y_{pw} = (1-\alpha)y_{pew}E$. In the steady state of the economy with a positive technological growth, $\frac{\Delta w}{w} = \frac{\Delta y_{pew}}{y_{pew}} + g$—wages grow at a constant rate equal to g once the economy hits the steady state. The rental price of capital, for our example, is $R = F_K(K, EL) = \alpha K^{\alpha-1}(EL)^{1-\alpha} = \alpha\frac{K^\alpha(EL)^{1-\alpha}}{K} = \alpha\frac{Y}{K} = \alpha\frac{Y/L}{K/L} = \alpha\frac{y_{pw}}{k_{pw}} = \alpha\frac{Y/EL}{K/EL} = \alpha\frac{y_{pew}}{k_{pew}}$. Thus, the rental price of capital will be constant for the steady-state economy with technological growth; it will be constant for the economy without technological progress as well. Since $r = R - \delta$, the real interest rate is predicted to be constant in the economies that reached their steady states. The (approximate) constancy of real interest rates is one of the facts of modern developed economies.

We can show, for this production function, that the shares of capital and labor income in total income are α and $1-\alpha$, respectively. For example, $\frac{RK}{Y} = \frac{\alpha\frac{Y}{K}K}{Y} = \alpha$. Along the balanced growth path, $w^*(t) = (1-\alpha)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} E(t) = (1-\alpha)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} E(0)(1+g)^t$, where $E(0)$ is the level of technology at the time the economy started operating.

Transitional dynamics. Assume the previous set-up and add labor-augmenting technological growth. Our per effective worker production function is $y_{pew} = k_{pew}^\alpha$. The steady state will happen when $s(k^*_{pew})^\alpha = (n+g+\delta)k^*_{pew}$, or when $k^*_{pew} = \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$. At the time of the change in the savings rate, $\frac{\Delta k_{pew}}{k_{pew}} = s'\frac{y^*_{pew}}{k^*_{pew}} - (n+g+\delta) = s'\frac{n+g+\delta}{s} - (n+g+\delta) > 0$ since $s' > s$. At the time of the change, the growth rate in capital per worker is $\frac{\Delta k_{pw}}{k_{pw}} = \frac{\Delta k_{pew}}{k_{pew}} + g = s'\frac{n+g+\delta}{s} - (n+\delta) > g$, the growth rate in total capital is $\frac{\Delta K}{K} = \frac{\Delta k_{pew}}{k_{pew}} + g + n = s'\frac{n+g+\delta}{s} - \delta > g + n$. The economy eventually will tend to its new steady state $k^{**}_{pew} = \left(\frac{s'}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$, when the growth rate in capital per effective worker (output per effective worker) is zero; the growth rate in capital per worker (output per worker) is g; and the growth rate in total capital (total output) is $g + n$.

Similarly to the previous example, before the change output per worker grew at the rate g, at the time of the change the growth rate jumped to $\alpha\frac{\Delta k_{pew}}{k_{pew}} + g = \alpha s'\frac{n+g+\delta}{s} - \alpha(n+g+\delta) + g > g$, and eventually it would start shrinking towards g. Total output, in turn, was growing at the rate $n + g$, the growth rate then jumped to $\alpha\frac{k_{pew}}{k_{pew}} + g + n = \alpha s'\frac{n+g+\delta}{s} - \alpha(n+g+\delta) + g + n > g + n$ at the time of the change, and it would start shrinking towards $g + n$, its balanced growth rate in the new steady state.