The Permanent Income Hypothesis (PIH)

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A 2-period formulation

- 2-period problem, periods 0 and 1.
- Within-period (instantaneous) utility function is quadratic:

\[ u(c_t) = -\frac{1}{2}(c_t - \bar{c})^2. \]

- Freely borrow/lend at the constant real interest rate \( r \).
- Endowments \( y_0 \) and \( y_1 \) are known at time 0.
- \( \bar{c} \) is the “bliss” consumption level. If \( c_t = \bar{c} \), a consumer attains the maximum utility possible, equal to 0.
- \( \bar{c} \geq c_t \) so that the marginal utility is positive: \( \bar{c} - c_t > 0 \).
- \( \beta \in (0, 1) \) is the time discount factor.
- \( \beta(1 + r) = 1. \)
Optimization problem

\[
\max_{c_0 \geq 0, c_1 \geq 0} U(c_0, c_1) = -\frac{1}{2}(c_0 - \bar{c})^2 - \beta \frac{1}{2}(c_1 - \bar{c})^2
\]

s.t. \( c_0 + \frac{c_1}{1 + r} = y_0 + \frac{y_1}{1 + r} \),

where \( \beta \in (0, 1) \) is the time discount factor. For this utility function,

\[
MU_0 = \bar{c} - c_0
\]

\[
MU_1 = \beta(\bar{c} - c_1).
\]
Optimum

At the optimum, the following two equations should be satisfied:

\[(\bar{c} - c^*_0) = \beta(1 + r)(\bar{c} - c^*_1)\]
\[c^*_0 + \frac{c^*_1}{1 + r} = y_0 + \frac{y_1}{1 + r}.\]

Since we assumed that \(\beta = \frac{1}{1+r}\), we can write the first of those equations as

\[\bar{c} - c^*_0 = \bar{c} - c^*_1, \quad \text{or} \quad c^*_0 = c^*_1.\]

Plugging this equilibrium condition into the second equation, we obtain \(c^*_0 + \frac{c^*_0}{1+r} = y_0 + \frac{y_1}{1+r}\), or

\[c^*_0 = c^*_1 = \frac{1 + r}{2 + r} \left( y_0 + \frac{y_1}{1 + r} \right).\]

Consumer, for these preferences, will prefer to smooth consumption across periods perfectly.
Infinite horizon–1

Assume instead that a consumer’s horizon is infinite, and s/he chooses consumption for periods \( t = 0, 1, 2, \ldots \). In this case,

\[
\max_{c_0 \geq 0, c_1 \geq 0, c_2 \geq 0, \ldots} U(c_0, c_1, c_2, \ldots) = -\frac{1}{2}(c_0 - \bar{c})^2 + \beta \left[ -\frac{1}{2}(c_1 - \bar{c})^2 \right] = u(c_0) \quad u(c_1) \\
+ \beta^2 \left[ -\frac{1}{2}(c_2 - \bar{c})^2 \right] = u(c_2) \quad u(c_3) \quad + \ldots \\
+ \beta^3 \left[ -\frac{1}{2}(c_3 - \bar{c})^2 \right] + \ldots \\
\text{s.t.} \quad c_0 + \frac{c_1}{1 + r} + \frac{c_2}{(1 + r)^2} + \frac{c_3}{(1 + r)^3} + \ldots = y_0 + \frac{y_1}{1 + r} + \frac{y_2}{(1 + r)^2} + \\
+ \frac{y_3}{(1 + r)^3} + \ldots
\]
More compactly,

\[
\max_{c_0 \geq 0, c_1 \geq 0, c_2 \geq 0, \ldots} U(c_0, c_1, c_2, \ldots) = \sum_{t=0}^{\infty} \left[ -\frac{1}{2} \beta^t (c_t - \bar{c})^2 \right]
\]

s.t. \[
\sum_{t=0}^{\infty} \frac{c_t}{(1 + r)^t} = \sum_{t=0}^{\infty} \frac{y_t}{(1 + r)^t}.
\]

Now, instead of finding just \(c_0^*\) and \(c_1^*\), we will need to find the whole (infinite) sequence \(\{c_0^*, c_1^*, c_2^*, \ldots\}\).

Not so hard! Just need the (optimality) Euler equations and the lifetime budget constraint.
The equations to be satisfied at the optimum

\[ MU_1 = (1 + r)MU_2 \]
\[ MU_2 = (1 + r)MU_3 \]
\[ MU_3 = (1 + r)MU_4 \]
\[ MU_4 = (1 + r)MU_5 \]
\[ \vdots \]

\[ \sum_{t=0}^{\infty} \frac{c^*_t}{(1 + r)^t} = \sum_{t=0}^{\infty} \frac{y_t}{(1 + r)^t}. \]
In terms of our utility function, the following equations should be satisfied at the optimum:

\[
\begin{align*}
\bar{c} - c_0^* &= (1 + r) \beta (\bar{c} - c_1^*) \\
MU_0 \\
\beta (\bar{c} - c_1^*) &= (1 + r) \beta^2 (\bar{c} - c_2^*) \\
MU_1 \\
\beta^2 (\bar{c} - c_2^*) &= (1 + r) \beta^3 (\bar{c} - c_3^*) \\
MU_2 \\
&\vdots \\
\sum_{t=0}^{\infty} \frac{c_t^*}{(1 + r)^t} &= \sum_{t=0}^{\infty} \frac{y_t}{(1 + r)^t}.
\end{align*}
\]

Since we assume that \( \beta = \frac{1}{1+r} \), the sequence of Euler equations implies

\[
c_0^* = c_1^*, \quad c_1^* = c_2^*, \quad c_2^* = c_3^* \ldots \Rightarrow c_0^* = c_1^* = c_2^* = c_3^* = \ldots = c^*.
\]
Plugging the result into the lifetime budget constraint,

\[ c^* \sum_{t=0}^{\infty} \frac{1}{(1 + r)^t} = \sum_{t=0}^{\infty} \frac{y_t}{(1 + r)^t}. \]

Note that \( \sum_{t=0}^{\infty} \frac{1}{(1 + r)^t} = 1 + \frac{1}{1 + r} + \frac{1}{(1 + r)^2} + \frac{1}{(1 + r)^3} + \ldots, \) and \( \frac{1}{1+r} < 1. \) We want to find \( S = 1 + x + x^2 + x^3 + \ldots, \) where \( x \equiv \frac{1}{1+r}. \) This sum will be equal to \( \frac{1}{1-x} = \frac{1}{1-\frac{1}{1+r}} = \frac{1+r}{r}. \)

\[ c^* = c_0^* = c_1^* = c_2^* = \ldots = \frac{r}{1+r} \left[ \sum_{t=0}^{\infty} \frac{y_t}{(1 + r)^t} \right] = y^p. \]

Milton Friedman: individual consumption in each period should be related to an estimate of the permanent income.
Aside

It is easy to show that

\[ S = 1 + x + x^2 + x^3 + x^4 + \ldots = \frac{1}{1 - x}, \text{ for } |x| < 1. \]

Multiply the LHS and RHS of the equation by \( x \),

\[ xS = x + x^2 + x^3 + x^4 + x^5 + \ldots, \]

and subtract the result from \( S \), to obtain

\[ S - xS = (1 + x + x^2 + x^3 + x^4 + \ldots) - (x + x^2 + x^3 + x^4 + \ldots) = 1. \]

Thus,

\[ S = \frac{1}{1 - x}. \]
Example: constant flow of endowments

If \( y_0 = y_1 = y_2 = \ldots = \bar{y} \), \( c^* \) will be equal to

\[
\frac{r}{1 + r} \bar{y} \left[ 1 + \frac{1}{1 + r} + \frac{1}{(1 + r)^2} + \frac{1}{(1 + r)^3} + \ldots \right] = \bar{y} \frac{r}{1 + r} \frac{1 + r}{r} = \bar{y}
\]
Stochastic incomes

- In reality, future incomes are uncertain (that is, stochastic). At time $t$, when making consumption decision for time $t$, we do not know for sure \( \{y_{t+1}, y_{t+2}, y_{t+3}, y_{t+4}, \ldots\} \).
- In this case, it does not make sense to set consumptions for periods \( c_{t+1}, c_{t+2}, \ldots \) once and for all, since new information about future incomes and permanent income will arrive in periods following $t$.
- The optimality (Euler) condition that links optimal consumptions in periods $t$ and $t + 1$, for the utility function we adopted, will read as:

\[
c_t^* = E_t(c_{t+1}^*),
\]

where $E_t(\cdot)$ is expectation conditional on information (about future endowments) available at time $t$. 
Stochastic Euler equation

Subtracting $c_t^*$ from both sides,

$$E_t \left( c_{t+1}^* - c_t^* \right) = E_t \Delta c_{t+1}^* = 0.$$  

It means that the expected future change in consumption, given all the available information at time $t$, is equal to zero, that is consumption does not change between periods $t$ and $t + 1$ if there is no additional information arriving between periods $t$ and $t + 1$ about consumer’s incomes. In statistics, a variable that has this property is called a **martingale**.

An implication of the martingale property of consumption is that consumption in period $t + 1$ will differ from consumption in period $t$ *only if* a consumer receives unexpected “news” about his permanent income.
Optimal consumption with stochastic incomes

In terms of the levels of consumption, we may derive the following relationship:

\[
c_t = y_t^p = E_t \left[ \frac{r}{1 + r} \left( y_t + \frac{y_{t+1}}{1 + r} + \frac{y_{t+2}}{(1 + r)^2} + \frac{y_{t+3}}{(1 + r)^3} + \ldots \right) \right].
\]
Important implications

- Consumption will change between adjacent periods if (a consumer’s estimate of) the permanent income changes.
- Consumption will adjust by a larger margin if an unexpected change in income is permanent (e.g., compare disability vs. short spell of unemployment).
- If the government contemplates about some policy affecting individual incomes (say, a tax cut) and wants to boost the economy via an increase in the aggregate consumption, it will only succeed if the policy affects permanent incomes a lot (say, a permanent reduction in income taxes). Otherwise, the reaction of consumers will be weak, if any.