ECON 385. Intermediate Macroeconomic Theory II. Solow Model With Technological Progress. Cobb-Douglas Example

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Equilibrium allocations

Let production function be of Cobb-Douglas type

$$Y = K^{\alpha}(EL)^{1-\alpha}.$$

It is CRS in K and L:

$$F(zK, zL, E) = (zK)^{\alpha}(EzL)^{1-\alpha} = z^{\alpha}z^{1-\alpha}\underbrace{K^{\alpha}(EL)^{1-\alpha}}_{=Y} = zY.$$

Define $z \equiv \frac{1}{EL}$ to obtain $y_{pew} = (k_{pew})^{\alpha}$, where $k_{pew} = \frac{K}{EL}$ and $y_{pew} = \frac{Y}{EL}$.

The steady-state equilibrium in this economy is defined from

$$s(k^*_{pew})^{\alpha} = (n + g + \delta)k^*_{pew}.$$

Thus,

$$k^*_{pew} = \left(\frac{s}{n + g + \delta}\right)^{\frac{1}{1-\alpha}}.$$

Furthermore,

$$y_{pew}^{*} = (k_{pew}^{*})^{\alpha} = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}},$$

$$c_{pew}^{*} = (1-s)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}.$$

Equilibrium prices

We know that

$$w = F_L = (1-\alpha)K^\alpha E^{1-\alpha}L^{-\alpha} = (1-\alpha)\frac{K^\alpha (EL)^{1-\alpha}}{L}$$
$$= (1-\alpha)\frac{Y}{L} = (1-\alpha)y_{pw} = (1-\alpha)y_{pew}E.$$

The rental price of capital

$$R = F_K = \alpha K^{\alpha-1}(EL)^{1-\alpha} = \alpha\frac{K^\alpha (EL)^{1-\alpha}}{K}$$
$$= \alpha\frac{Y}{K} = \alpha\frac{Y/L}{K/L} = \alpha\frac{y_{pw}}{k_{pw}} = \alpha\frac{Y/(EL)}{K/(EL)} = \alpha\frac{y_{pew}}{k_{pew}}.$$

The real interest rate is equal to

$$r = F_K - \delta = \alpha\frac{y_{pew}}{k_{pew}} - \delta.$$

Prices in the steady state equilibrium

In the steady-state equilibrium, $w^*(t) = (1-\alpha)y^*_{pew}E(t)$ and $R^* = \alpha\frac{y^*_{pew}}{k^*_{pew}}$.
For our example,

$$w^*(t) = (1-\alpha)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} E(0)(1+g)^t,$$

$$R^* = \alpha\frac{\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}}{\left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}} = \alpha\left(\frac{s}{n+g+\delta}\right)^{-1} = \alpha\frac{n+g+\delta}{s},$$

and

$$r^* = \alpha\frac{n+g+\delta}{s} - \delta.$$

Growth rates in the steady state—1

Note that k^*_{pew} and y^*_{pew} are constant in the steady state. What about K and Y, and k_{pw} and y_{pw}? By definition,

$$K = k_{pew} EL.$$

Therefore,

$$\frac{\Delta K}{K} = \frac{\Delta k_{pew}}{k_{pew}} + \frac{\Delta E}{E} + \frac{\Delta L}{L}.$$

In the steady state, $\frac{\Delta k^*_{pew}}{k^*_{pew}} = 0$ and so

$$\frac{\Delta K}{K} = \frac{\Delta E}{E} + \frac{\Delta L}{L} = g + n$$

Aggregate capital grows at a constant rate equal to $(g + n)$. The same can be shown for aggregate output, Y.

Growth rates in the steady state—2

y_{pw} and k_{pw} will grow in the steady state at the rate g.

$$k_{pw} = k_{pew}E.$$

Therefore,

$$\frac{\Delta k_{pw}}{k_{pw}} = \frac{\Delta k_{pew}}{k_{pew}} + \frac{\Delta E}{E}.$$

In the steady state, $\frac{\Delta k^*_{pew}}{k^*_{pew}} = 0$ and so

$$\frac{\Delta k_{pw}}{k_{pw}} = \frac{\Delta E}{E} = g$$

Capital per worker grows at a constant rate equal to g. The same can be shown for the growth rate of output per worker, y_{pw}.

Golden rule steady state

If $Y = K^{\alpha}(EL)^{1-\alpha}$, then the golden-rule savings rate in the economy is equal to α, the share of capital income in total income.

For the economy with technological progress, the golden-rule capital per worker is obtained from $MPK(k^*_{gold}) = n + g + \delta$.

For this production function,
$MPK = F_K(K, L) = \alpha K^{\alpha-1}(EL)^{1-\alpha} = \alpha \left(\frac{K}{EL}\right)^{\alpha-1} = \alpha k_{pew}^{\alpha-1}$.

Thus, the golden rule capital per worker is obtained from

$$\alpha k_{pew,gold}^{\alpha-1} = n + g + \delta.$$

and so $k_{pew,gold} = \left(\frac{\alpha}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$.

Golden rule steady state—2

If the economy saves its capital income, the total savings in the economy are αY, the per effective worker savings are αy_{pew}.

For this economy, the steady state occurs when

$$\alpha(k^*_{pew})^\alpha = (n + g + \delta)k^*_{pew},$$

i.e., when

$$k^*_{pew} = \left(\frac{\alpha}{n + g + \delta}\right)^{\frac{1}{1-\alpha}},$$

which is exactly equal to the golden rule capital per worker we've just found.