ECON 385. Intermediate Macroeconomic Theory II. Solow Model Without Technological Progress. Cobb-Douglas Example

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Equilibrium allocations

Let production function be of Cobb-Douglas type

$$Y = K^\alpha L^{1-\alpha}.$$

It is CRS:

$$F(zK, zL) = (zK)^\alpha (zL)^{1-\alpha} = z^{1-\alpha+\alpha} K^\alpha L^{1-\alpha} = z \underbrace{K^\alpha L^{1-\alpha}}_{=Y} = zY.$$

Define $z \equiv \frac{1}{L}$ to obtain $y_{pw} = (k_{pw})^\alpha$, where $k_{pw} = \frac{K}{L}$ and $y_{pw} = \frac{Y}{L}$.

The steady-state equilibrium in this economy is defined from

$$s(k^*_{pw})^\alpha = (n+\delta)k^*_{pw}.$$

Thus,

$$k^*_{pw} = \left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}}.$$

Furthermore,

$$y^*_{pw} = (k^*_{pw})^{\alpha} = \left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}},$$

$$c^*_{pw} = (1-s)\left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}.$$

Countries with higher investment rates, lower population growth rates, lower depreciation rates, and lower degree of the decline in the marginal product of capital (higher α) will be richer relative to the otherwise similar countries.

Equilibrium prices

We know that

$$w = F_L = (1-\alpha)K^\alpha L^{-\alpha} = (1-\alpha)\frac{K^\alpha L^{1-\alpha}}{L}$$
$$= (1-\alpha)\frac{Y}{L} = (1-\alpha)y_{pw}.$$

The rental price of capital

$$R = F_K = \alpha K^{\alpha-1}L^{1-\alpha} = \alpha\frac{K^\alpha L^{1-\alpha}}{K} = \alpha\frac{Y}{K} = \alpha\frac{Y/L}{K/L} = \alpha\frac{y_{pw}}{k_{pw}}.$$

The real interest rate is equal to

$$r = F_K - \delta = \alpha\frac{y_{pw}}{k_{pw}} - \delta.$$

Prices in the steady state equilibrium

In the steady-state equilibrium, $w^* = (1-\alpha)y^*_{pw}$ and
$R^* = \alpha\frac{y^*_{pw}}{k^*_{pw}}$.
For our example,

$$w^* = (1-\alpha)\left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}},$$

$$R^* = \alpha\frac{\left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}}{\left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}}} = \alpha\left(\frac{s}{n+\delta}\right)^{-1} = \alpha\frac{n+\delta}{s},$$

and

$$r^* = \alpha\frac{n+\delta}{s} - \delta.$$

Income shares of labor and capital costs

We know that $Y = wL + RK$, and the share of labor costs/income and capital costs/income in total income are equal to $\frac{wL}{Y}$ and $\frac{RK}{Y}$, respectively.
Thus,

$$\frac{wL}{Y} = \frac{wL/L}{Y/L} = \frac{w}{y_{pw}} = \frac{(1-\alpha)y_{pw}}{y_{pw}} = 1-\alpha$$

$$\frac{RK}{Y} = \frac{RK/L}{Y/L} = \frac{Rk_{pw}}{y_{pw}} = \alpha\frac{y_{pw}}{k_{pw}}\frac{k_{pw}}{y_{pw}} = \alpha.$$

Thus, for a Cobb-Douglas production function $Y = K^{\alpha}L^{1-\alpha}$, the share of labor and capital costs in total income are constant, and equal to $(1-\alpha)$ and α, respectively.

Growth rates in the steady state

Note that k^*_{pw} and y^*_{pw} are constant in the steady state. What about K and Y? By definition,

$$K = k_{pw}L.$$

Therefore,

$$\frac{\Delta K}{K} = \frac{\Delta k_{pw}}{k_{pw}} + \frac{\Delta L}{L}.$$

In the steady state, $\frac{\Delta k^*_{pw}}{k^*_{pw}} = 0$ and so $\frac{\Delta K}{K} = \frac{\Delta L}{L} = n$—aggregate capital grows at a constant rate equal to the growth rate in population, n. The same can be shown for aggregate output, Y.

Golden rule steady state—1

If $Y = K^{\alpha}L^{1-\alpha}$, then the golden-rule savings rate in the economy is equal to α, the share of capital income in total income.

In other words, if the economy saves at the rate $s = \alpha$, then it will reach the golden-rule steady state.

For the economy without technological progress, the golden-rule capital per worker is obtained from $MPK = n + \delta$.

For this production function, $MPK = F_K(K, L) = \alpha K^{\alpha-1}L^{1-\alpha} = \alpha \left(\frac{K}{L}\right)^{\alpha-1} = \alpha k_{pw}^{\alpha-1}$.

Thus, the golden rule capital per worker is obtained from

$$\alpha k_{pw,gold}^{\alpha-1} = n + \delta.$$

and so $k_{pw,gold} = \left(\frac{\alpha}{n+\delta}\right)^{\frac{1}{1-\alpha}}$.

Golden rule steady state—2

If the economy saves its capital income, the total savings in the economy are αY, the per worker savings are αy_{pw}.

For this economy, the steady state occurs when $\alpha (k^*_{pw})^\alpha = (n+\delta) k^*_{pw}$, i.e., when $k^*_{pw} = \left(\frac{\alpha}{n+\delta}\right)^{\frac{1}{1-\alpha}}$, which is exactly equal to the golden rule capital per worker we've just found.