ECON 385. Intermediate Macroeconomic Theory II. Solow Model Without Technological Progress

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Solow Growth Model

- A major paradigm:
  –widely used in policy making
  –benchmark against which most recent growth theories are compared
- Looks at the determinants of economic growth and the standard of living in the long run
- Readings: Mankiw and Scarth, 4th edition, Chapter 7

Basic Assumptions

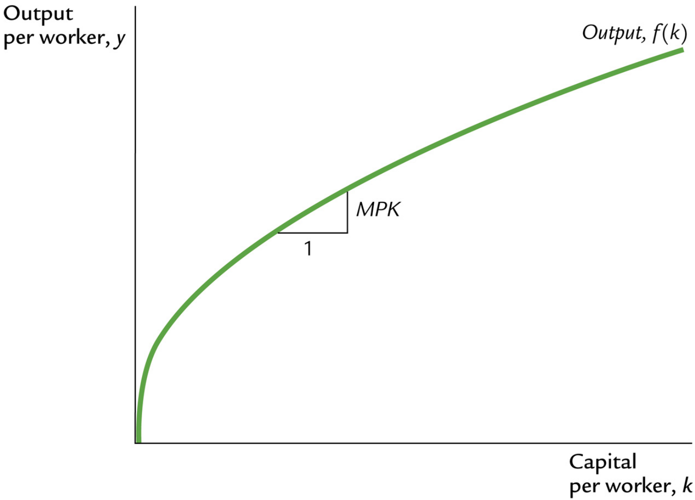
- Competitive firms maximise profits
- Produce homogeneous output (Y) using neoclassical production function
- Production factors (K, L) may *grow* over time
- Technological progress assumed exogenous; technology is a **public** good (non-excludable, and non-rival)

Neoclassical Production Function

- In aggregate terms: $Y = AF(K, L)$
- Assume technology constant A and normalized to 1
- Define: $y = Y/L =$ output per worker and $k = K/L =$ capital per worker
- Assume constant returns to scale: $zY = F(zK, zL)$ for any $z > 0$
- Pick $z = 1/L$. Then

$$
\begin{aligned}
Y/L &= F(K/L, 1) \\
y &= F(k, 1) \\
y &= f(k), \quad \text{where} \quad f(k) = F(k, 1)
\end{aligned}
$$

The production function



National Income Identity

$$\underbrace{\text{Y}}_{\text{supply}} = \underbrace{\text{C} + \text{I}}_{\text{demand}}$$

- Closed economy: NX = 0
- No government: G = 0
- In per worker terms ($c = C/L$ and $i = I/L$)

$$y = c + i$$

The Consumption Function

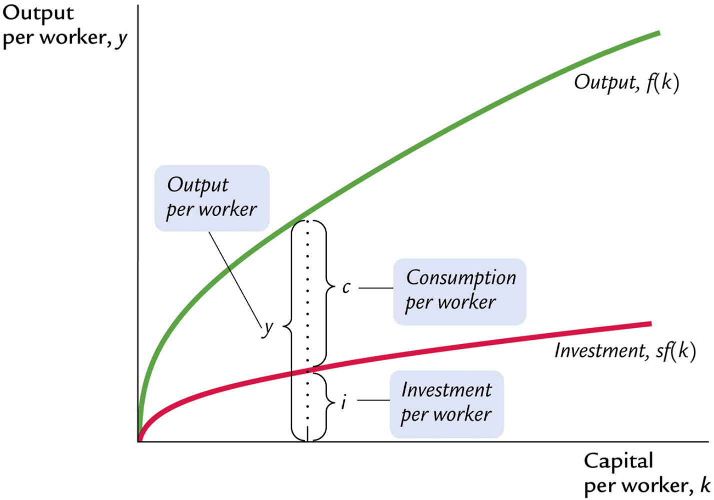
- $0 < s < 1$= the saving rate, the fraction of income that is saved (s is an **exogenous** parameter)
- Consumption per worker:

$$c = (1 - s)y$$

Investment and Saving

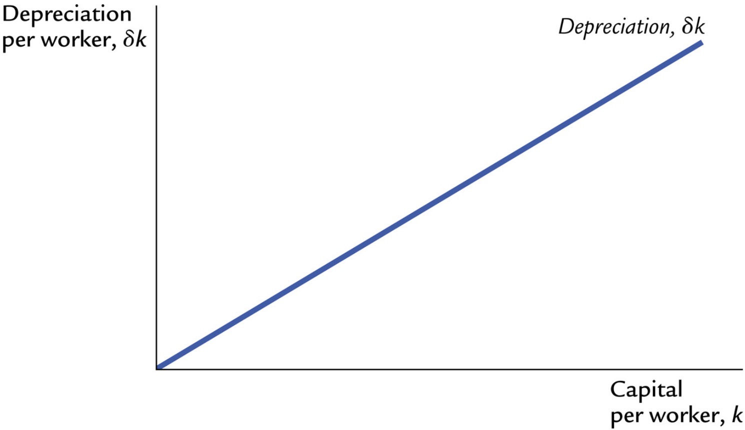
- Saving (per worker)$=sy$ (by definition)
- National income identity is $y = c + i$
- Rearrange to get:
  $i = y - c = y - (1-s)y = sy$
  investment = saving (Say's Law)
- Using the results above,

$$i = sy = sf(k)$$


Output
per worker, y
Output, f(k)
Output
per worker
c
Consumption
per worker
y
Investment, sf(k)
i
Investment
per worker
Capital
per worker, k

How is the capital per worker determined?

- New capital is added each period by adding investment to the old stock of capital
- A portion of old capital wears off in the production process which leads to a lower capital stock. The process of “losing” capital in the process of production is called **depreciation**.
- Let depreciation rate be δ. E.g., $\delta = 0.1$ means that each year 10% of capital per worker is wears off in production process.


Depreciation
per worker, δk
Depreciation, δk
Capital
per worker, k

Capital Accumulation

- Change in capital stock = investment – depreciation

$$\Delta k = i - \delta k$$

Since $i = sf(k)$, this becomes:

$$\underbrace{\Delta k = sf(k) - \delta k}_{\textbf{fundamental equation of the Solow model}}$$

The Law of Motion for k

$$\underbrace{\Delta k = sf(k) - \delta k}_{\textbf{fundamental equation of the Solow model}}$$

- Determines behavior of capital, k, over time
- which, in turn, determines behavior of all of the other endogenous variables because they all depend on k.
- E.g., income per person: $y = f(k)$
  consumption per person: $c = (1 - s)f(k)$

The Steady State

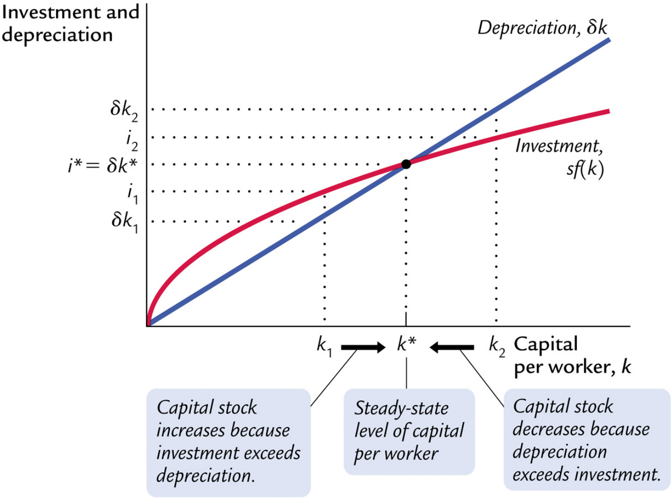
- If investment is just enough to cover depreciation

$$sf(k^*) = \delta k^*$$

then capital per worker will remain constant

$$\Delta k = 0$$

- This constant value, denoted k^*, is called the **steady state capital stock**



Investment and depreciation
Depreciation, δk
δk2
i2
i* = δk*
i1
δk1
Investment, sf(k)
k1
k*
k2
Capital per worker, k
Capital stock increases because investment exceeds depreciation.
Steady-state level of capital per worker
Capital stock decreases because depreciation exceeds investment.

Notes

- ★ Steady-state (SS) level of capital per worker k^* is the one economy gravitates to in the long run regardless of its initial level of capital per worker be it above k^*, or below k^*.
- ★ At k^*, we can determine the SS (long-run) value the long-run values of consumption per worker (c^*), and investment per worker, y^*.
- ★ At SS, output per worker, y^*, and therefore the standard of living stays the same over time.
- ★ With zero population and technological growth, the growth rate of total output at the SS is zero!

Numerical Example

Let production function be
$Y = F(K, L) = K^{1/2}L^{1/2}$. Let $s = 0.3$, and
$\delta = 0.1$.

In per capita terms, $Y/L = (K^{1/2}L^{1/2})/L$. And so
$y = K^{1/2}L^{-1/2}$, or $y = (\frac{K}{L})^{1/2} = k^{1/2}$.

The law of motion of k:
$\Delta k = sk^{1/2} - \delta k = 0.3k^{1/2} - 0.1k$.

At the SS, $\Delta k = 0$. Thus, k^* solves:
$0.3(k^*)^{1/2} - 0.1k^* = 0$. And so
$k^*/(k^*)^{1/2} = 0.3/0.1 = 3$. Thus, $k^* = 3^2 = 9$.

$y^* = (k^*)^{1/2} = 3$; $c^* = (1 - s)y^* = 0.7 \times 3 = 2.1$

Numerical example, step-by-step

Assumptions: $y = \sqrt{k}$; $s = 0.3$; $\delta = 0.1$; initial $k = 4.0$

Year	k	y	c	i	δk	Δk
1	4.000	2.000	1.400	0.600	0.400	0.200
2	4.200	2.049	1.435	0.615	0.420	0.195
3	4.395	2.096	1.467	0.629	0.440	0.189
4	4.584	2.141	1.499	0.642	0.458	0.184
5	4.768	2.184	1.529	0.655	0.477	0.178
⋮						
10	5.602	2.367	1.657	0.710	0.560	0.150
⋮						
25	7.321	2.706	1.894	0.812	0.732	0.080
⋮						
100	8.962	2.994	2.096	0.898	0.896	0.002
⋮						
∞	9.000	3.000	2.100	0.900	0.900	0.000

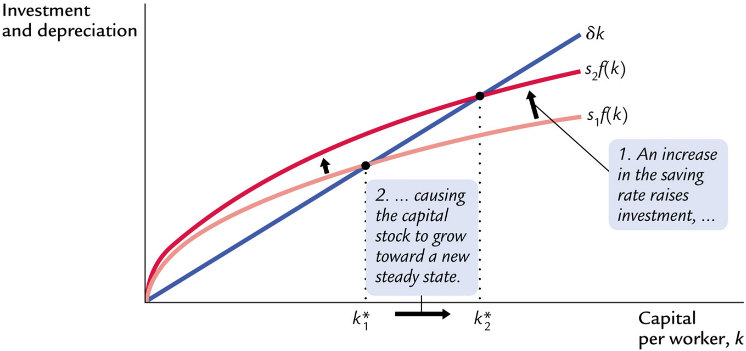
Table 7-2 Approaching the Steady State: A Numerical Example
Mankiw and Scarth: Macroeconomics, Canadian Fifth Edition

Comparative statics and **dynamics**

Comparative statics: how one SS compares to another SS when we change one of the **exogenous** parameters, e.g., s, or δ

Comparative dynamics: how the economy moves from one SS to another when we change one of the **exogenous** parameters, e.g., s, or δ

- Example: Change in savings rate: $s_1 \rightarrow s_2, s_2 > s_1$


Investment and depreciation
δk
s2f(k)
s1f(k)
1. An increase in the saving rate raises investment, ...
2. ... causing the capital stock to grow toward a new steady state.
k*1
k*2
Capital per worker, k

Predictions of the Solow Model

- Higher $s \Rightarrow$ higher k^*
- And since $y = f(k)$, higher $k^* \Rightarrow$ higher y^*
- Thus, the Solow model predicts that countries with higher rates of saving and investment will have higher levels of capital and income per worker in the long run.
- What about the data?

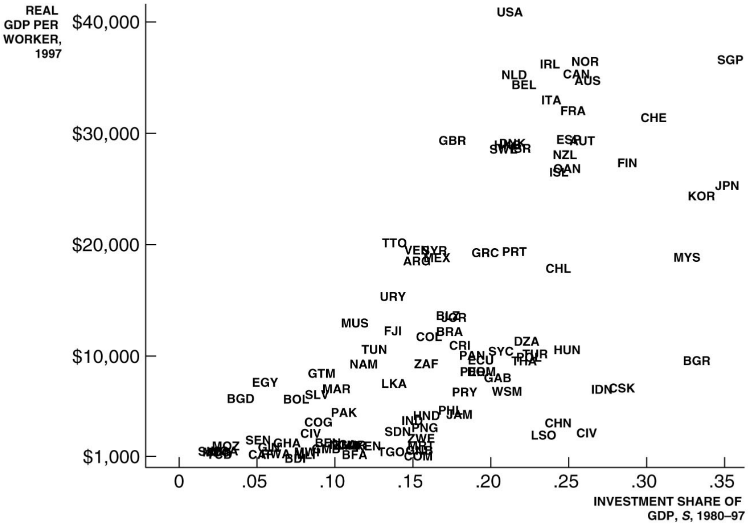


FIGURE 2.6 GDP PER WORKER VERSUS THE INVESTMENT RATE

Economic Growth, 2nd Edition

Investment Rates and Income per Person

- Strong correlation
- What determines savings?
  - tax policy
  - financial markets
  - culture/preferences
  - institutions (Acemoglu. Why nations fail?)

The Golden Rule Savings Rate

- Different values of s lead to different steady states. How do we know which is the "best" steady state? (normative issue)
- Economic well-being depends on consumption, so the "best" steady state has the highest possible value of consumption per person: $c^* = (1-s)f(k^*)$
- An increase in s
  –leads to higher k^* and y^*, which may raise c^*
  –reduces consumption share of income $(1-s)$, which may lower c^*
- So, how do we find the s and k^* that maximize c^*?

Golden Rule of Capital

- Planner wants to maximize $c = y - i = f(k) - sf(k)$. At the SS, $c^* = f(k^*) - \delta k^*$ since $sf(k^*) = \delta k^*$ at the SS.
- Different saving rates, s, will give different values of k^* so we can write

$$c^* = c^*(s) = f(k^*(s)) - \delta k^*(s)$$
$$\Rightarrow \frac{\partial c^*}{\partial s} = \frac{\partial k^*}{\partial s}\left[f'(k^*) - \delta\right] \quad \text{by chain rule}$$

- Low values of $s \Rightarrow$ low values of $k^* \Rightarrow$ high values of MPK, and increasing consumption with s, the reverse is true for high values of s.

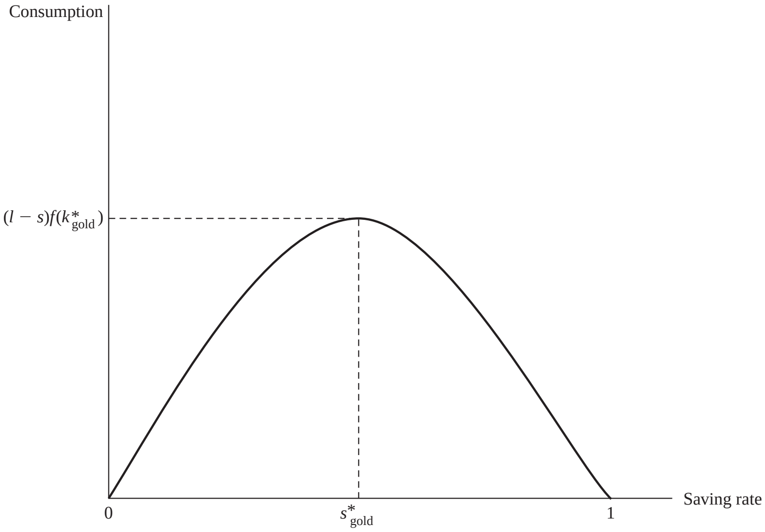


FIGURE 2.6 The golden rule level of saving rate, which maximizes steady-state consumption.

Source: Acemoglu. Introduction to Modern Economic Growth.

Assumptions: $y = \sqrt{k}$; $\delta = 0.1$

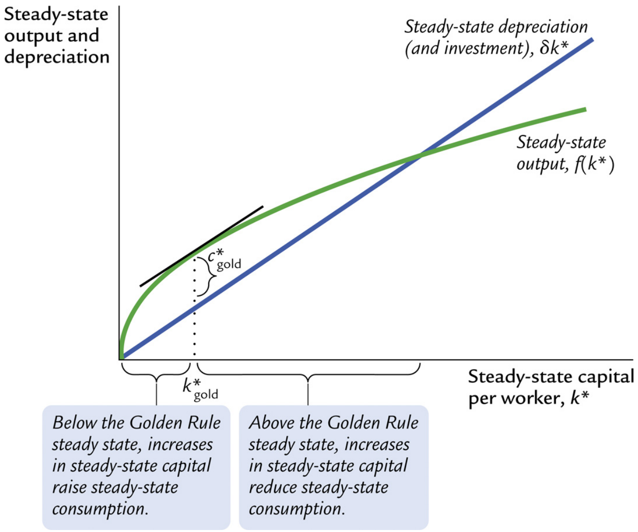
s	k^*	y^*	δk^*	c^*	MPK	$MPK - \delta$
0.0	0.0	0.0	0.0	0.0	•	•
0.1	1.0	1.0	0.1	0.9	0.500	0.400
0.2	4.0	2.0	0.4	1.6	0.250	0.150
0.3	9.0	3.0	0.9	2.1	0.167	0.067
0.4	16.0	4.0	1.6	2.4	0.125	0.025
0.5	**25.0**	**5.0**	**2.5**	**2.5**	**0.100**	**0.000**
0.6	36.0	6.0	3.6	2.4	0.083	–0.017
0.7	49.0	7.0	4.9	2.1	0.071	–0.029
0.8	64.0	8.0	6.4	1.6	0.062	–0.038
0.9	81.0	9.0	8.1	0.9	0.056	–0.044
1.0	100.0	10.0	10.0	0.0	0.050	–0.050

Golden rule savings rate

Maximum consumption per capita achieved when

$$\frac{\partial c^*}{\partial s} = \frac{\partial k^*}{\partial s}\left[f'(k^*) - \delta\right] = 0$$
$$\Rightarrow \underbrace{f'(k^*)}_{=\text{MPK}} = \delta,$$

when the slope of the production function equals the slope of the depreciation line.

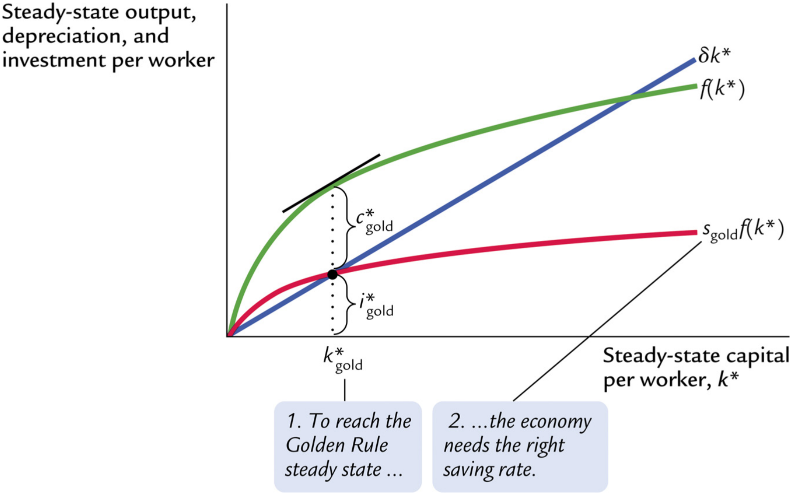


Steady-state output and depreciation
Steady-state depreciation (and investment), δk*
Steady-state output, f(k*)
c*gold
k*gold
Steady-state capital per worker, k*
Below the Golden Rule steady state, increases in steady-state capital raise steady-state consumption.
Above the Golden Rule steady state, increases in steady-state capital reduce steady-state consumption.

Numerical Example, previous numbers but s

The planner needs to "induce"/set the saving rate s that will support k^*_{gold}.

- Find k^*_{gold} and s_{gold}.
- At the SS: $s(k^*_{\text{gold}})^{1/2} = 0.1k^*_{\text{gold}}$. Thus, $s = 0.1 \times (k^*_{\text{gold}})^{1/2}$ (1).
- We also know that $\text{MPK}(k^*_{\text{gold}}) = \delta$.
- $\text{MPK} = f'(k)$. How to find $f'(k)$? For a power function, $f(x) = x^\alpha$, $f'(x) = \alpha x^{\alpha-1}$.
- Thus for our example $\text{MPK} = 1/2(k^*_{\text{gold}})^{-1/2}$, or $1/2 \times 1/(\sqrt{k^*_{\text{gold}}})$. And so... $\sqrt{k^*_{\text{gold}}} = 5$, and $k^*_{\text{gold}} = 25$. From (1), $s_{\text{gold}} = 0.1 \times 5 = 0.5$.


Steady-state output, depreciation, and investment per worker
δk*
f(k*)
c*gold
sgoldf(k*)
i*gold
k*gold
Steady-state capital per worker, k*
1. To reach the Golden Rule steady state ...
2. ...the economy needs the right saving rate.

The transition to the GR Steady State

- The economy does **NOT** have a tendency to move toward the Golden Rule steady state
- Achieving the Golden Rule requires that policymakers adjust s
- This adjustment leads to a **new steady state** with higher consumption
- But what happens to consumption during the transition to the Golden Rule?

Starting with too little k and small s, raise s at t_0

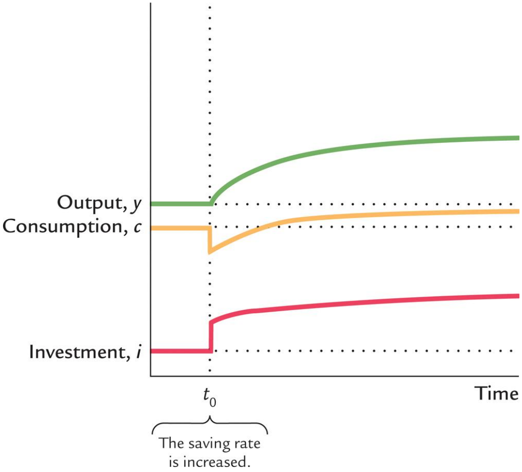


Figure 7-10 Increasing Saving When Starting With Less Capital Than in the Golden Rule Steady State
Mankiw and Scarth: Macroeconomics, Canadian Fifth Edition

Starting with too large k and large s, lower s at t_0

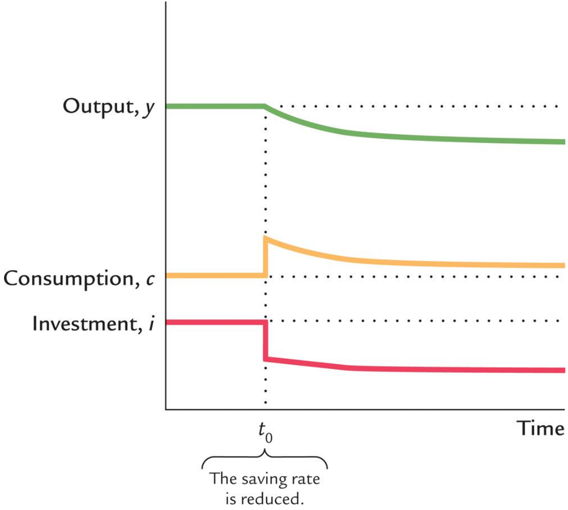


Figure 7-9 Reducing Saving When Starting With More Capital Than in the Golden Rule Steady State
Mankiw and Scarth: Macroeconomics, Canadian Fifth Edition

Relaxing the assumption of no population growth

Assume that population in the economy grows $n\%$ per year, that is, $\frac{\Delta L}{L} = n$. The law of motion of aggregate capital $\Delta K = I - \delta K = sY - \delta K$.

Note that $k = \frac{K}{L}$. Using the math note I sent you, we can show

$$\begin{aligned}
\frac{\Delta k}{k} &= \frac{\Delta K}{K} - \frac{\Delta L}{L} \\
&= \frac{I - \delta K}{K} - \frac{\Delta L}{L} \\
&= s\frac{Y}{K} - \delta - \frac{\Delta L}{L} \\
&= s\frac{Y/L}{K/L} - \delta - \frac{\Delta L}{L} \\
&= s\frac{y}{k} - \delta - n.
\end{aligned}$$

Population growth

$$\frac{\Delta k}{k} = s\frac{y}{k} - \delta - n$$

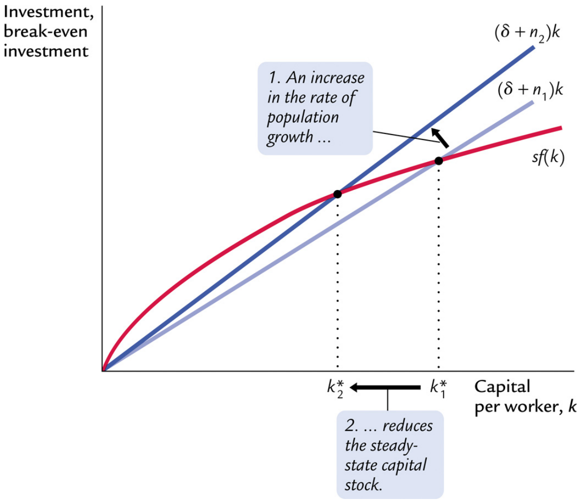
Multiplying both sides of the equation by k, we obtain:

$$\Delta k = sy - \delta k = \underbrace{sy}_{\text{actual investment}} - \underbrace{(n+\delta)k}_{\text{break-even investment}}$$

Break-even investment is the amount of investment necessary to keep k constant. It includes:

- δk to replace capital as it wears out
- nk to equip new workers with capital (otherwise, k would fall as the existing capital stock would be spread more thinly over a larger population of workers)

Population growth changes from $n_1 = 0$ to $n_2 > 0$



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- What about the data?

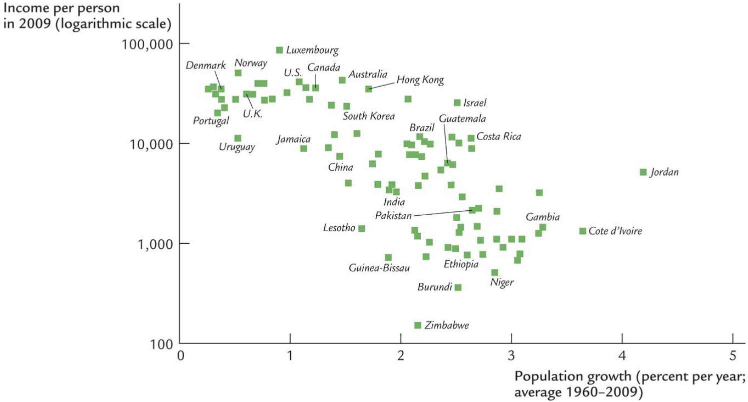

Figure 7-13 International Evidence on Population Growth and Income per Person
Mankiw and Scarth: Macroeconomics, Canadian Fifth Edition

The Golden Rule with Population Growth

$$c^* = y^* - i^* = f(k^*) - (\delta + n)k^*$$

c^* is maximized when

$$\text{MPK} = n + \delta$$

Summary of Predictions of Solow Model

Solow growth model shows that, in the long run, a country's standard of living, y, depends

- positively on its saving rate (s)
- negatively on its population growth rate (n)

Change in policies ($\uparrow s$ or $\downarrow n$) result in

- higher output per capita **level** in the long run
- faster growth temporarily
- but not everlasting growth of per capita income (since in steady state nothing changes)

Next

- Solow model augmented with technological progress
- Policies to promote growth
- Convergence